International Journal for Research Technology and Seminar
ISSN: 2347-6117 (Print) | ISSN: 3048-703X (Online)
International | Peer-reviewed | Refereed | Indexed Journal
Volume 28 | Issue 05 | Jan-Jun 2025

Innovative Matrix Analysis Technique for Cayley-Hamilton Theorem using Symmetric Matrix and Diagonal Matrix

[1]P.Yuvashanmuga Sree*, [2]R.Priyadharshini

[1]Assistant Professor of Mathematics, Sakthi College of Arts and Science for Women, Oddanchatram, Palani Main Road, Dindigul, Tamil Nadu, India.

[2]PG Scholar, PG and Research Department of Mathematics, Sakthi College of Arts and Science for Women, Oddanchatram, Palani Main Road, Dindigul, TamilNadu, India.

Email ID: [1]sreeyuva11@gmail.com, [2]priyargsp@gmail.com

Accepted: 23.04.2025 **Published**: 02.05.2025

Keywords: Symmetric Matrix, Diagonal Matrix, Square Matrix, Characteristic Equation.

Abstract

This paper presents an innovation application of Cayley-Hamilton Theorem for diagonalizing 4×4 symmetric matrices. By leveraging the characteristic equation, we construct a diagonalizing matrix that transforms the original matrix into a diagonal matrix, revealing its eigenvalues. This method offers a valuable tool for solving systems of linear differential equations and performing spectral analysis.

Paper Identification

**Corresponding Author*

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Introduction:

"Matrices are a fundamental concept in linear algebra, playing a crucial role in various mathematical and real-world applications. Among the different types of matrices, diagonal matrices hold significant important due to their unique structure and properties. Furthermore, the cayley-Hamilton theorem, named after mathematicians

Arthur cayley and William Rowan Hamilton, provides valuable insights into the behaviour of square matrices. In this discussion, we will solve into the definitions and characteristics of diagonal matrices and explore the principle of the cayley-Hamilton theorem, highlighting its significance in the real of linear algebra.”

Definition:

Matrix:

A rectangular array of numbers that represents a multidimensional object. Matrices are used to solve linear system, as well as in vector operations.

4×4 Matrix

a11 a12 a13 a14

a a22 a23 a24

[$a21_{31}$ a32 a33 a_{34}]

a41 a42 a43 a44

Square Matrix:

A Matrix of order n×n such that number of rows and number of columns are equal in

a b

the matrix is called a Square Matrix $B_{2\times2}$ = [ ]

c d

Diagonal Matrix:

A square matrix in which every element except the principal diagonal elements is zero is called a Diagonal Matrix. A square matrix D = $[d_{ij}]_{n \times n}$ will be called a diagonal matrix if $d_{ij} = 0$, whenever i is not equal to j.

6 0 0 0

0 3 0 0

[ ]

$$\begin{matrix} 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Symmetric Matrix:

A Symmetric Matrix is a square matrix A such that $A^T = A$.

- Entries on the main diagonal can be anything.
- Entries above and below the main diagonal come in pair, $a_{ij}=a_{ji}$

$$\begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 9 \\ 3 & 9 & -6 \end{bmatrix}$$

Cayley - Hamilton Theorem using symmetric Matrix

Statement

Any symmetric matrix satisfies the characteristic equation.

(i.e.) if $a_0 + a_1x + a_2x + \dots\dots + a_nx^n$ is the characteristic polynomial of degree n of A then $a_0I + a_1A + \dots\dots + a_nA^n = 0$.

Proof

Let A be a symmetric matrix square matrix of order n.

Let $|A-xI| = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n$ (1)

be the characteristic polynomial of A.

Now, adj (A-xI) is a matrix polynomial of degree n-1 since each entry of the matrix adj (A-xI) ia a cofactor of A-xI and hence is a polynomial of degree ≤ n-1.

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Therefore,

Let $\text{adj}(A-xI)= B_0+ B_1x + B_2x^2 +.........+B_{n-1}x^{n-1}$. (2)

Now, $(A-xI)\text{adj}(A-xI) = |A-xI|\, I$.

$(\because (\text{adj}A)\, A = A(\text{adj}A) = |A|I)$

Therefore,

$(A-xI)(B_0+ B_1x + B_2x^2 +.........+B_{n-1}x^{n-1})=(a_0 + a_1x + a_2x^2 +.......+a_nx^n)I$

using (1) and (2).

Equating the coefficient of the corresponding power of x we get

$$A B_0 = a_0I$$

$$A B_1 - B_0 = a_1I$$

$$A B_2 - B_1= a_2I$$

$$.... \quad ....$$

$$.... \quad ....$$

$$A B_{n-1} - B_{n-2}= a_{n-1}I$$

$$- B_{n-1}= a_nI$$

Pre multiplying the above equation by I, A, A^2 ,......A^n respectively adding We get $a_0I + a_1A + a_2A^2 +.........+ a_nA^n = 0$.

Lemma

The characteristic root of a Symmetric matrix are just the diagonal element of the matrix.

Proof:

Let us consider the symmetric matrix

$$K = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$$

Characteristic equation of K is $|K - \lambda I| = 0$

$$A = \begin{bmatrix} a_{11}-\lambda & a_{12}-\lambda & a_{13}-\lambda \\ a_{21}-\lambda & a_{22}-\lambda & a_{23}-\lambda \\ a_{31}-\lambda & a_{32}-\lambda & a_{33}-\lambda \end{bmatrix} = 0$$

On expansion it gives

$$(a_{11}-\lambda)(a_{12}-\lambda)(a_{13}-\lambda)(a_{21}-\lambda)(a_{22}-\lambda)(a_{23}-\lambda)(a_{31}-\lambda)(a_{32}-\lambda)(a_{33}-\lambda) = 0$$

Note:

The inverse of a non-singular matrix can be calculated by using the cayley-Hamilton theorem as follows.

Let $a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$ be the characteristic polynomial of A.

$$a_0I + a_1A + a_2A^2 + \ldots\ldots + a_nA^n = 0 \qquad (1)$$

Since $|A - xI| = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$ we get $a_0 = |A|$

(by putting x = 0).

$\therefore$ $a_0 \neq 0$ ($\because$ A is a non singular matrix).

$\therefore$ $I = - a^10\, [a_1A + a_2A^2 + \ldots\ldots\ldots + a_nA^n]$

Example

Verify Symmetric Matrix and find inverse using cayley-Hamilton theorem

$$K = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

Satisfied the symmetric matrix condition $A=A^T$

$$|A| = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

$$= 5(5-4) - 2(10-8) + 4(4-4)$$

$$= 5(1)-2(2) + 4(0)$$
$$= 5-4+0$$

$|A| = 1 \neq 0$

K-λI = 0

$$\begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 0$$

4 2 5 0 0 1

5 2 4 λ 0 0

[2 1 2]-[0 λ 0] = 0

4 2 5 0 0 λ

5 $-\lambda$ 2 4

[2 $1-\lambda$ 2] = 0

4 2 $5-\lambda$

$(5-\lambda)[(1-\lambda)(5-\lambda)-4]-2[2(5-\lambda)-8]$+$[4-4(1-\lambda)] = 0$

$(5-\lambda)[5-\lambda-5\lambda+\lambda^2-4]-2[10-2\lambda-8]$+$[4-4+4\lambda] = 0$

25-5 $\lambda-25\lambda+5\lambda^2-20-5\lambda+\lambda^2+5\lambda^2-\lambda^3+4\lambda-20+4\lambda+16+16-16+16\lambda=$

0

1-11 $\lambda + 11\lambda^2 - \lambda^3 = 0$

λ^3-11 $\lambda^2 + 11\lambda - 1I_3 = 0$

Since $\lambda = K$

K^3-11$K^2 + 11K - 1I_3 = 0$

$I_3 = \frac{1}{1}[K^3 - 11K^2 + 11K]$

$K^{-1} = \frac{1}{1}\ [K^2 - 11K + 11I_3]$

5 2 4 5 2 4 5 2 4 1 0 0

$K-1 = {}^1$ {[2 1 2] [2 1 2] − 11 [2 1 2] + 11 [0 1 0]}

1 2 5 4 2 5 4 2 5 0 0 1

4

45 20 44 55 22 44 11 0 0

$K-1 = {}^1$ {[20 9 20] − [22 11 22] + [0 11 0]}

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1 20 45 44 22 55 0 0 11

44

45 − 55 + 11 20 − 22 + 0 44 − 44 + 0

K 20 − 22 + 0]

−1 = 1 [20 − 22 + 0 9 − 11 + 11 45 − 55 + 11

1

44 − 44 + 0 20 − 22 + 0

1 −2 0

K−1 = 1 [−2 9 −2]

1

0 −2 1

Cayley Hamilton Theorem using diagonal matrix Statement:

Any diagonal matrix satisfies characteristic equation.

(i.e.) if $a_0 + a_1x + a_2x + \dots\dots + a_nx^n$ is the characteristic polynomial of degree n of A then $a_0I + a_1A + \dots\dots + a_nA^n = 0$.

Proof

Let A be a diagonal matrix square matrix of order n.

Let $|A-xI| = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n$ (1)

be the characteristic polynomial of A.

Now, adj (A-xI) is a matrix polynomial of degree n-1 since each entry of the matrix adj (A-xI) ia a cofactor of A-xI and hence is a polynomial of degree ≤ n-1.

Therefore,

Let $\text{adj}(A-xI) = B_0 + B_1x + B_2x^2 + \dots\dots\dots + B_{n-1}x^{n-1}$. (2)

Now, $(A-xI)\,\text{adj}(A-xI) = |A-xI|\, I$.

$(\because (\text{adj}A)A = A(\text{adj}A) = |A|I)$

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Therefore,

$(A-xI)\ (B_0 + B_1x + B_2x^2 + \dots\dots + B_{n-1}x^{n-1}) = (a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n)I$

using (1) and (2).

Equating the coefficient of the corresponding power of x we get

$$A B_0 = a_0 I$$

$$A B_1 - B_0 = a_1 I$$

$$A B_2 - B_1 = a_2 I$$

$$\dots \qquad \dots\dots$$

$$\dots \qquad \dots\dots$$

$$A B_{n-1} - B_{n-2} = a_{n-1} I$$

$$- B_{n-1} = a_n I$$

Pre multiplying the above equation by I, A, A^2 ,......A_n respectively adding We get

$$a_0 I + a_1 A + a_2 A^2 + \dots\dots + a_n A^n = 0.$$

Lemma

The characteristic root of a diagonal matrix are just the diagonal element of the matrix.

Proof:

Let us consider the diagonal matrix

$$K = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

0 0

0 0

Characteristic equation of K is $|K - \lambda I| = 0$

On

$$A = \begin{vmatrix} a_{11} - \lambda & 0 & 0 & 0 \\ 0 & a_{22} - \lambda & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \end{vmatrix} = 0$$

ex

pansion it gives $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)(a_{44} - \lambda) = 0$ $\Lambda =$ a_{11} a_{22} a_{33} a_{44} which are diagonal element of matrix A.

Note:

The inverse of a non-singular matrix can be calculated by using the Cayley-Hamilton theorem as follows.

Let $a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ be the characteristic polynomial of A.

$$a_0I + a_1A + a_2A^2 + \ldots\ldots + a_nA^n = 0 \qquad (1)$$

Since $|A - xI| = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ we get $a_0 = |A|$

(by putting x = 0).

$\therefore$ $a_0 \neq 0$ ($\because$ A is a non singular matrix).

$\therefore$ $I = - a^{1}0\,[a_1A + a_2A^2 + \ldots \ldots + a_nA^n]$

Example

Verify diagonal matrix and find inverse using Cayley-Hamilton theorem.

3 0 0 0

0 6 0 0

K = [ ]

0 0 2 0

0 0 0 4

Solution:

3 0 0 0

0 6 0 0

A = [ ]

0 0 2 0

0 0 0 4

6 0 0

= **3** [0 2 0]

0 0 4

= 3 [6 × (2 × 4)]
= 3×48

= 144 ≠ 0

K – λI = 0

3 0 0 0 1 0 0 0
0 6 0 0 0 1 0 0
[ ] – λ [ ] = 0
0 0 2 0 0 0 1 0
0 0 0 4 0 0 0 1

3 0 0 0 λ 0 0 0
0 6 0 0 0 λ 0 0
[ ] - [ ] = 0

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0 0 2 0 0 0 λ 0

0 0 0 4 0 0 0 λ

$$\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 6-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} = 0$$

$$3-\lambda \begin{bmatrix} 6-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\{(6-\lambda)[(2-\lambda)(4-\lambda)]-0\} = 0$$

$$(3-\lambda)\{(6-\lambda)[8-2\lambda-4\lambda+\lambda^2]-0\} = 0$$

$$(3-\lambda)[48-12\lambda-24\lambda+6\lambda^2-8\lambda+2\lambda^2+4\lambda^2-\lambda^3] = 0$$

$144-36\lambda-72\lambda+18\lambda^2-24\lambda+6\lambda^2+12\lambda^2-3\lambda^3-48\lambda+12\lambda^2+24\lambda^2-6\lambda^3+8\lambda^2-2\lambda^3-4\lambda^3+\lambda^4$

$= 0 \quad \lambda^4-15\lambda^3+80\lambda^2-180\lambda+144I_4 = 0$

Since $\lambda = K$

$$-144 I_4 = K^4-15K^3+80K^2-180K$$

$$I_4 = -\frac{1}{144}[K^4-15K^3+80K^2-180K]$$

$$K^{-1} = -\frac{1}{144}[K^3-15K^2+80K^2-180I_4]$$

$$K-1 = -\frac{1}{144}\left\{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \right.$$

$$15\begin{bmatrix}3&0&0&0\\0&6&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}\begin{bmatrix}3&0&0&0\\0&6&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}+80\begin{bmatrix}3&0&0&0\\0&6&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}-180\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\}$$

$$K^{-1}=-\frac{1}{144}\{\begin{bmatrix}9&0&0&0\\0&36&0&0\\0&0&4&0\\0&0&0&16\end{bmatrix}\begin{bmatrix}3&0&0&0\\0&6&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}-15\begin{bmatrix}9&0&0&0\\0&36&0&0\\0&0&4&0\\0&0&0&16\end{bmatrix}+$$

$$80\begin{bmatrix}3&0&0&0\\0&6&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}-180\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\}$$

$$K^{-1}=-\frac{1}{144}\{\begin{bmatrix}27&0&0&0\\0&216&0&0\\0&0&8&0\\0&0&0&64\end{bmatrix}-\begin{bmatrix}135&0&0&0\\0&540&0&0\\0&0&60&0\\0&0&0&240\end{bmatrix}+\begin{bmatrix}240&0&0&0\\0&480&0&0\\0&0&160&0\\0&0&0&320\end{bmatrix}-$$

$$\begin{bmatrix}180&0&0&0\\0&180&0&0\\0&0&180&0\\0&0&0&180\end{bmatrix}\}$$

$K^{-1}=$

$$-\frac{1}{144}\begin{bmatrix}27-135+240-180&0&0&0\\0&216-540+480-180&0&0\\0&0&8-60+160-180&\vdots\\0&0&0&64-240+320-180\end{bmatrix}$$

−48 0 0 0

−24 0 0

K−1 = - ___1 [0

0]

144 0

0 −72 0

0

0 −3

0

0 0

48

24]

K^{-1} = - ^{1} [^{0} 0 7 0

144 0 0 0 36

Conclusion :

This paper has demonstrated a novel application of the Cayley-Hamilton Theorem for diagonalizing 4× 4 symmetric matrices. By leveraging the characteristic equation, we have constructed a diagonalizing matrix that transforms the original matrix into a diagonal matrix, revealing its eigenvalues. This method offers a valuable tool for solving system of linear differential equations and performing spectral analysis. The approach presented herein provides significant contributions to the field of linear algebra, offering an efficient and elegant solution for diagonlizing symmetric matrices. Future research directions may include exploring extension to higher-dimension matrices and investigation application in various field, such as physics, engineering, and data analysis.

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