COMPUTABLE FIELD EMBEDDINGS AND DIFFERENCE CLOSED FIELDS: A MATHEMATICAL ANALYSIS

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Abstract

Computable fields are the primary focus of this paper, which contributes to the compelling field hypothesis. If the elements of a structure's area are linked with regular numbers, then the operations that take place in this space are also computable functions. Maps between fields can be expanded to maps between their mathematical terminations, according to a variety of classical results. We're thinking about when this will be possible. That is, should there exist a computable ex-pressure to the arithmetical terminations if the majority of the fields contained are computable and we are provided a computable guide? The classical theorems hold effectively in a computable field F if certain requirements are met, which are both important and sufficient. It's possible to apply our findings to fields that have been shown to contain automorphisms, which are referred to as difference fields. This paper investigates how to successfully incorporate difference fields within computed difference-shut fields (these are existentially shut difference fields, to be talked about). For computable difference fields, even the most innocent resemblance to remarkable consequences fails in every way. A large range of fields (including abelian augmentations of a prime field) can be found in which the corresponding consequence can be better understood.

Paper Identification

Introduction

If $\mathcal{F}$ is a field, a polynomial $\mathcal{F}[X]$ is called separable if it has no repeated roots an element a E of an algebraic field extension $E\backslash\mathcal{F}$ is called separable over $\mathcal{F}$ if its minimal polynomial over $\mathcal{F}$ is a separable polynomial. An algebraic field extension $E\backslash\mathcal{F}$ is called separable if every element of ∈ is separable over$\mathcal{F}$. Recall that if $\mathcal{F}$ is finite or characteristic zero, then it is perfect, i.e., every algebraic extension is a separable extension. An algebraic field extension$E\backslash\mathcal{F}$ is called purely inseparable if E\F contains no separable elements. Equivalently, ∈ is a field of characteristic p >0 and every element of ∈is the unique root of a polynomial

X^{pn} − a =0 with a∈F. Given an algebraic field extension $E\backslash F$, the set

$$\mathcal{F}^s = \{a \in \mathcal{E} : a \text{ is separable over } \mathcal{F}\}$$

Is the maximal separable extension of $\mathcal{F}$ inside of ∈and is called the separable closure of $\mathcal{F}$ in∈. The field extension $\in\backslash\mathcal{F}^s$ is purely inseparable. In the special case where ∈ $\mathcal{F}$ is the algebraic closure of $\mathcal{F}$, $\mathcal{F}^s$ is called the separable closure of $\mathcal{F}$ and is the maximal separable extension of$\mathcal{F}$. An algebraic field extension $\in\backslash\mathcal{F}$ is normal if every irreducible polynomial in $\mathcal{F}$[X] that has a root in E factors completely in ∈[X]. Normal separable extensions E\F is called a Galois extension and has associated to it the Galois group Gal ($\in\backslash\mathcal{F}$) of automorphisms of E Fixing $\mathcal{F}$. Recall that the Galois group obeys the fundamental theorem of Galois Theory: the normal subgroups H Gal ($\in\backslash\mathcal{F}$) correspond to the intermediate normal field extensions.

Computable Fields

Recall that the splitting set $S_{\mathcal{F}}$ of $\mathcal{F}$ is the set of all polynomials p$\mathcal{E}\mathcal{F}$[X] which is reducible over$\mathcal{F}$. The splitting set of a field is not necessarily computable (see but it is always c.e. If the splitting set of$\mathcal{F}$ is computable, then we say that $\mathcal{F}$ has a splitting algorithm. Finite fields and algebraically closed fields trivially have splitting algorithms. That Q has a splitting algorithm, and also that many other field extensions also have a splitting algorithm:

Theorem: The field Q has a splitting algorithm. If a computable field $\mathcal{F}$ has a splitting algorithm, and is transcendental over$\mathcal{F}$, then $\mathcal{F}$(a) has a splitting algorithm if a separable and algebraic over$\mathcal{F}$, then $\mathcal{F}$ (a) has a splitting algorithm. Moreover, the splitting algorithm for $\mathcal{F}$ (a) is uniform in the minimal polynomial for an over$\mathcal{F}$given a field $\mathcal{F}$ and an element a is either transcendental over$\mathcal{F}$, or separable and algebraic over $\mathcal{F}$, we know that $\mathcal{F}$ (a) has a splitting algorithm. However, the algorithm depends on whether is transcendental or algebraic. To find a splitting algorithm uniformly, we must know which the case is. That every computable field F has a computable algebraic closure $\mathcal{F}$, and moreover there is a computable embedding $\tau\mathcal{F}\to\mathcal{F}$. We call such anembedding a Rabin embedding. Moreover, he characterized the image of F under this embedding:

Theorem: Let $\mathcal{F}$ is a computable field. Then algebraically closed field $\mathcal{F}$ and a computable field embedding $\tau\mathcal{F}$ → gebraic over $\tau(\mathcal{F})$. Moreover, for any such $\mathcal{F}$ andτ, the image $\tau(\mathcal{F})$ equivalent to the splitting set of $\mathcal{F}$There is a computable $\mathcal{F}$ such that $\mathcal{F}$ is al-off $\mathcal{F}$ in$\mathcal{F}$ is Turing$A$ computable field $\mathcal{F}$ has a dependence algorithm if given a and $b_1...b_n$, we can compute whether a is algebraically independent over $b_1...b_n$.A field has a dependence algorithm if and only if it has a computable transcendence base (see, for example. In particular, fields of finite transcendence degree have a dependence algorithm. Convention By an extension $\epsilon\backslash\mathcal{F}$ of computable fields, we mean that there is a computable embedding of $\mathcal{F}$ intoϵ.

Difference Fields

Difference fields were first studied by Ritt in the 1930s. A good reference on the classical algebraic theory of difference fields is the book. A difference field is a field $\mathcal{F}$ together with an embedding σ$\mathcal{F}\to\mathcal{F}$. If σ is onto, ($\mathcal{F}$, σ) is called inversive. As every difference field has a unique inversive closure up to isomorphism, we lose nothing by assuming that all of our difference fields are inversive.

A difference field ($\mathcal{F}$, σ) is called a difference closed field if it is existentially closed in the language of difference fields. Difference closed fields arose in the model theoretic study of difference fields. $\mathcal{F}$ Is difference closed if and only if?

σ is an-automorphism of $\mathcal{F}$;

$\mathcal{F}$ Is algebraically closed;

$$V \subseteq U \times \sigma(U)$$

The condition (iii) may be viewed as saying that certain systems of equations and in equations have solutions in F. Conditions (i), (ii), and (iii) axiomatize the theory ACF A of difference closed fields. ACFA$_P$ is decidable, and moreover the theories ACF A_p of difference closed fields of characteristic *p* are also decidable for any *p*, including *p*. ACF A is the model companion of the theory of difference fields and hence every formula is equivalent, modulo ACF A, to an existential formula.

Thus, we have: Every computable difference closed field has a computable (full) elementary diagram. We call a structure with a computable elementary diagram decidable; thus every difference closed field is decidable.

Extending Embeddings into the Algebraic Closure

We begin by showing that if $\mathcal{F}$ is any computable field with a splitting algorithm, τF $\rightarrow \mathcal{F}$ is a Rabin embedding, and α$\mathcal{F} \rightarrow \kappa$ is a computable embedding of $\mathcal{F}$ into an algebraically closed field κ, then there is a computable embedding of $\mathcal{F}$ into $\mathcal{F}$ extending α. In particular, the new results here are in the case of characteristic p>0. The new issue we have to deal with in characteristic p>0 is that Theorem 2.1 fails for non-separable extensions. We begin by finding the separable closure of a field F within its algebraic closure$\mathcal{F}$.

Lemma Let $\mathcal{F}$ is a computable field. Then the separable closure of $\mathcal{F}$ is c.e. If $\mathcal{F}$has a splitting algorithm, then the separable closure $\mathcal{F}^s$ of $\mathcal{F}$ in $\mathcal{F}$ is computable (so that have a plitting algorithm).

Proof. Embed F in its algebraic closure$\mathcal{F}$. An element a $\mathcal{F}$ is separable if and only if there is a polynomial p(X) $\mathcal{F}$[X] of degree m with p (a) 0 and with m distinct roots in$\mathcal{F}$. Thus the separable closure of $\mathcal{F}$ is c.e. If $\mathcal{F}$ has a splitting algorithm, then given $\alpha \in \mathcal{F}$ we can find the minimal polynomial p of an over$\mathcal{F}$. Then $\alpha$is separable over if and only if p has no repeated roots, which happens if and only if p (a) = 0. (Here, p'(X) is the derivative of ($\mathcal{F}$) with respect to, treating the coefficients as constants.) So separable closure of $\mathcal{F}$ is computable we are now ready to extend an embedding from a field with a splitting algorithm. The main idea is to break the embedding into two steps; first to extend an embedding $\infty\mathcal{F} \rightarrow \kappa$ to an embedding β$\mathcal{F}^s \rightarrow \kappa$ of the separable closure of $\mathcal{F}$ into$K$ and second to note that β extends to a unique embedding of$\mathcal{F}$ into $\kappa$and that this extension is computable from β.

Theorem: let $\mathcal{F}$ be a computable field and $\iota\mathcal{F} \rightarrow \mathcal{F}$ a Rabin embedding of $\mathcal{F}$ into its algebraic closure. Suppose that $\mathcal{F}$ has a splitting algorithm. Then there is a Turing functional Φ such that whenever α $\mathcal{F} \rightarrow$ is an embedding of $\mathcal{F}$ into an algebraically closed fieldκ, $\Phi^{\alpha K}\mathcal{F} \rightarrow \kappa$ is an embedding of $\mathcal{F}$ into $\kappa\iota$-extending α.

Proof. Since $\mathcal{F}$ has a splitting algorithm, the image $\iota(\mathcal{F})$ of κ in $\mathcal{F}$ is computable. We may identify $\mathcal{F}$ with its image. By Lemma 3.1 the separable closure $\mathcal{F}^s$ of $\mathcal{F}$ is computable as a subset of $\mathcal{F}$ and has a splitting algorithm.

Let κ be an algebraically closed field and α: F $\rightarrow \kappa$ a field embedding. We will begin Extend α to an embedding βF$^s \rightarrow \kappa$. Leta, a . . .} be by describing a procedure to s an enumeration of the elements F. Start with β defined only on $\mathcal{F}$ and ı-extending α.

Using the splitting algorithm for F, find the minimal polynomial $P_1\mathcal{F}[X]$ of a_1 over $\mathcal{F}$. Find a solution $b_1\kappa$ to α (P_1). Then define β on $\mathcal{F}(a_1)$ by mapping a_1 to b_1. Since a_1 is algebraic and separable over F (and we know its minimal polynomial), we have a splitting algorithm for $\mathcal{F}$ (a_1). The separable closure of $\mathcal{F}(a_1)$ is $\mathcal{F}^s$. Now find the minimal polynomial $P_2\mathcal{F}[X]$ of a_2 over $\mathcal{F}(a_1)$, and a solution b_2 to α (P_2). Define β on $\mathcal{F}(a_1, a_2)$ by mapping a_2 to b_2. Note that a_2 is separable over $\mathcal{F}(a_1)$ since

$$\mathcal{F} \subseteq \mathcal{F}(a_1) \subseteq \mathcal{F}(a_1, a_2) \subseteq \mathcal{F}^s$$

And $\mathcal{F}^s$ is a separable algebraic extension of $\mathcal{F}$. Since a_2 is algebraic and separable over $\mathcal{F}(a_1)$, we have a splitting algorithm for $\mathcal{F}$ (a_1, a_2). Its separable closure is still F^s. Continuing in this way, we define and embedding β $\mathcal{F}^s \rightarrow \kappa$ which ı-extends α $\mathcal{F} \rightarrow \kappa$.

In characteristic zero, we are done since $\mathcal{F}^s\mathcal{F}$. In characteristic p0, we can extend β to an embedding $\mathcal{F} \rightarrow \kappa$ in the following manner. Given b$\mathcal{F}$, find the minimal polynomial P $\mathcal{F}^s[X]$ of b over $\mathcal{F}^s$ (recalling that F^s have a splitting algorithm). Then P (X) is of the form $X^{pn} - r$ 0 with r$\mathcal{F}$. Note that b is the unique solution of p(X) 0, and we can find the unique solution c to β (p) (X) 0. Map b to c. This is the unique embedding of $\mathcal{F}$ into κ extending β.

The construction was uniform in α and κ and so we get the desired Turing functional Φ.

We are now ready to prove Theorem 1.1, which says that a field has a splitting algorithm if and only if it has the computable (or uniform) extend-ability of embeddings property.

Proof of Theorem 1 the implication (1) (2) is Theorem 3 The implication (2) (3) is immediate. It remains to show the implication (3) (1).

Fix $\mathcal{F} \rightarrow \mathcal{F}$, a computable embedding of $\mathcal{F}$ into a computable presentation $\mathcal{F}$ of its algebraic closure. Suppose that every computable embedding of $\mathcal{F}$ into a computable algebraically closed field κι-extends to a computable embedding of $\mathcal{F}$ into κ.

We will attempt to construct a computable field κ and a computable embedding α$\mathcal{F} \rightarrow \kappa$ while attempting to diagonalize against all potential computable extensions $\varphi_e\, \mathcal{F} \rightarrow \kappa$ (by having α(a) φ_e(ı(a)) for some a $\mathcal{F}$). We know that the construction must fail, and from this we will conclude that F has a splitting algorithm.

We construct κ by an effective Henkin-style construction. The Henkin construction will be similar to one that can be used to prove Rabin's theorem that every field embeds into a computable presentation of its algebraic closure. See, for example, where this construction is carried out in reverse mathematics. (Rabin's original proof constructed the algebraic closure using a quotient of a polynomial ring with infinitely many variables.) Let L_F be the language of fields with constant symbols for the elements of $\mathcal{F}$, and let T be the consistent theory of algebraically closed fields together with the atomic diagram of $\mathcal{F}$. By quantifier elimination for the theory of algebraically closed fields, T is a complete theory and hence is decidable. We want to construct a decidable prime model of the theory T, which gives an algebraic closure κ of $\mathcal{F}$ together with an embedding of $\mathcal{F}$ into κ. The embedding α $\mathcal{F} \rightarrow \kappa$ will be built as part of the Henkin construction. Constructing a prime model requires a slight modification of the Henkin construction, which is possible in this case—we must also omit the type of an element that is transcendental over κ for the general theorem on effectively omitting types).

Let C = {c_0, c_1 . . .} be the new constant symbols for the Henkin construction. The domain of K will be the equivalence classes of some computable equivalence relation on C. Let $\varphi_e\mathcal{F} \rightarrow C$ be a list of partial computable functions which we interpret as the possible computable embeddings $\mathcal{F}\rightarrow\kappa$. Let {$a_0$, a_1, a_2 . . .} be a computable enumeration of$\mathcal{F}$. We use $\underline{a}_i$ to denote the constant symbol associated with $a_i\mathcal{F}$.

Construction At each stage s, we define formulas δ_0. . . δ_s in the language $L_{\cup C}$ which form the partial diagram of κ at stage s. The theory = {δ_0, δ_1 . . .} will be a complete theory extending T which is the complete diagram of the model κ (with the domain of κ being the equivalence classes in C by the equivalence relation c d c = d). At stage s, let $\psi_s = \delta_0\mathcal{F}\ \delta_{s1}$. We can arrange the construction so that the only constant symbols from $\mathcal{F}$ that appear in δ_s are $\underline{a}_0$. . . $\underline{a}_s$.

At stage 0, let δ_0 be $c_0 = c_0$. κ

At stage s = 4t + 1, we try to diagonalize against a φ_e for e ≤ t. Search for an e ≤ t and an i<s+5 such that $\varphi_{e,t}(\imath(a_i)) = c_j$ and (where c=(c_0, c_1, . . .) is the sequence of constants from C that appear in ψ_s):

By $\psi_s[\bar{x}\sim\bar{c}]$, we mean that the variables $\bar{x}$ = (x_0, x_1, . . .) have been substituted for the constants $\bar{c}$ = (c_0, c_1, . . .). This is a bounded search since T is decidable and we only have to search through finitely many $\underline{a}_i$. If such an e exists, choose the least e such that we have not yet diagonal zed against φ_e. Then set δ_s to be the formula $\underline{a}_i \neq c_j$ for that e. If no such e exists, set δ_s to be the formula $c_0 = c_0$.

At stages s = 4t + 2, s = 4t + 3 and s = 4t + 4, we act as in the standard method of constructing a computable prime model (e.g., Theorems 5.1 and 7.1 of Harizanov's survey, as follows: is of the form (x) φ (x), then

At stage s = 4t + 2, we add a Henkin witness for δ_t. If δ_t

Let c_i be a constant which does not appear in ψ_s and let δ_s be $\varphi(c_i)$. Otherwise, set δ_s to be the formula $c_0 = c_0$.

At stage s = 4t+3, we satisfy the completeness requirement for the sentence χ_t from some fixed listing $(\chi_t)_{t\omega}$ of the sentences in the language $L_{\cup C}$. Let $\bar{c}$ be the constants from C which appear in ψ_s and χ_t. Check whether if this is the case, let δ_s be χ_t. Otherwise, let δ_s be $\neg\chi_t$.

At stage s = 4t+4, we omit the type of an element transcendental over$\mathcal{F}$. We will have c_t satisfy some polynomial over$\mathcal{F}$Let $\bar{c}$ be the constants from C which appear in ψ_s, except for c_t. Search for a polynomial $p(x)\mathcal{F}[X]$ such that Set δ_s to be the formula $p(c_t) = 0$. Some such polynomial p must exist as the type of a transcendental over $\mathcal{F}$ is a non-principal type. φ

Verification By the standard Henkin construction arguments, we get a decidable prime model $\mathcal{K}$ whose domain consists of equivalence classes from C. We get a computable embedding α of $\mathcal{F}$ into $\mathcal{K}$ by mapping $\alpha\mathcal{F}$ to the element of $\mathcal{K}$labeled by the symbolα. Then α extends to an embedding β of $\mathcal{F}$ into$\mathcal{K}$, which we may represent as a computable map $\varphi e\ \mathcal{F}\rightarrow C$ (by, say, choosing φe(a) to be the least element of C in the equivalence class of β(a), which we can do computably since the equivalence classes are computable).There is a stage s0 after which we never diagonalize against an e′< e. We never diagonalize against e.

Claim. Let $b\in\mathcal{F}$ and t be a stage such that $\varphi_{e,t}(b)\downarrow= c_j$ for some jεω. Let s = 4t + 1. Then b∈> τ(F) if and only if there is some i such that

$$T \vdash \forall\bar{x}(\psi_s[\bar{x}/\bar{c}] \Rightarrow \underline{a}_i = x_j).$$

Proof. Given (∗), in K the constant symbol a_i is interpreted as the equivalence class of c_j . Thus α maps a_i to the equivalence class of c_j since β extends α and is one-to-one, $\tau(a_i)$ = b. On the other hand, suppose that b τ(F), say b = a_i, and suppose to the contrary that (∗) does not hold. We have two cases. First, if i < s + 5, then we set δ_s to be the formula $a_i \neq c_j$. Then $\alpha(a_i) \neq c_j = \varphi_e(\imath(a_i))$, which is a contradiction. Second, if $i \geq s+5$, then let s>s be the first stage of the form s=4t+1 with i<s+5. We have i>s (as if i≤s we could have chosen s − 4). Since the only constant symbols from F that appear in ψ_s are $\underline{a}_0$..... $\underline{a}_s'$, and i> s, $\underline{a}_i$ does not appear in ψ_s. Then we have we set δ_s' to be the formula $\underline{a}_i \neq c_j$ which again yields a contradiction.

Hence (∗) holds, The claim gives us a decision procedure for $\tau(*)\mathcal{F}$. At any stage s, there are only finitely many constants c ∈C mentioned in ψ_s, and hence only finitely many a_i such that we might possibly have (∗). So given b∈ $\mathcal{F}$, compute s = 4t+1 ≥ s_0 and j such that φ b ↓= c_j,$\mathcal{F}$ e,t$^{(*)}$ and then check (∗) for the finitely many possible a_i to decide whether b$\tau(\mathcal{F})$.

Corollary Let $\mathcal{F}$ be a computable algebraic field and $\mathcal{F}\to\mathcal{F}$ a computable embed-ding of $\mathcal{F}$into a computable presentation of its algebraic closure. Let κ be a computable algebraically closed field.

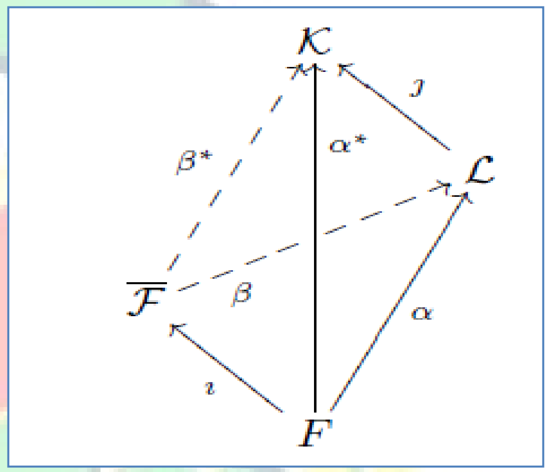
Then the following are equivalent:

$\mathcal{F}$ Has a splitting algorithm, There is a Turing functional Φ which takes an embedding α∈ $\mathcal{F}\to\kappa$ to an embedding Φ^α of ⊓into κ extending α, Every computable embedding of every computable embedding of into κ ı-extends to a computable embedding of into κ Proof. By Theorem 1.1, it success to show that in the statement implies that $\mathcal{F}$ has the computable extend-ability property with respect to ı$\mathcal{F}\to\mathcal{F}$.

Let α∈ $\mathcal{F}\to L$ be a computable embedding of $\mathcal{F}$into a computable algebraically closed field L.

We can enumerate, in$L$, the algebraic closure of the prime field and this contains the image α ($\mathcal{F}$). So we may assume that L is the algebraic closure of its prime field.

We can compute an embedding $L\to\kappa$ and let α ($\mathcal{F}$) $\to\kappa$ be α. By (3), there is an embedding β α ($\mathcal{F}$) → K which ı-extends β



$\beta:\overline{\mathcal{F}} \to \mathcal{L}$

Conclusion

We will conclude this paper by applying our results to difference closed fields. The main idea will be to note that ($\mathcal{F}$, σ) embeds into a computable difference closed field if and only if there is an embedding ı of $\mathcal{F}$ into $\mathcal{F}$and an automorphism τ of $\mathcal{F}$ such that τ ı-extends σ. In the one direction, this will follow from an effective Henkin construction, while on the other hand it will follow from the fact that the algebraic closure of the prime field can be enumerated in any difference closed field.

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