EXACT SOLUTIONS IN QUANTUM MECHANICS: VIM RECONSTRUCTION AND BEYOND

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Abstract

The Schrödinger equation stands as a cornerstone in the realm of quantum mechanics, governing the behavior of quantum particles and waves. This work presents a comprehensive reconstruction of the Variational Iteration Method (VIM) for the precise determination of exact solutions to both linear and nonlinear Schrödinger equations. In the linear Schrödinger context, we revisit and extend the VIM approach, providing a systematic framework for obtaining closed-form solutions that capture the quantum behavior of non-interacting particles. This reconstruction not only enhances the efficiency and accuracy of existing methods but also unveils new insights into the mathematical structure of the linear Schrödinger equation. Furthermore, we explore the formidable domain of nonlinear Schrödinger equations, where the interaction between quantum particles gives rise to rich and complex phenomena. Through the reconstructed VIM, we offer a powerful tool to tackle these nonlinear challenges, enabling the discovery of exact solutions that were previously elusive. This advancement contributes to our understanding of the intricate interplay between nonlinearity, dispersion, and soliton dynamics in quantum systems. The reconstructed VIM methodology is demonstrated through a series of illustrative examples, showcasing its applicability across various physical scenarios. It offers a valuable resource for researchers in quantum mechanics, mathematical physics, and related fields, enabling them to navigate the intricacies of linear and nonlinear Schrödinger equations with precision and insight. This study not only refines existing techniques but also paves the way for further advancements in the exploration of quantum phenomena through mathematical modeling and analysis.

Paper Identification

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Introduction

Differential equations are frequently used to discuss physics difficulties. Sometimes, there just isn't a foolproof answer to things like this. In addition, numerical methods make computational analysis of these solutions much easier, saving both time and money compared to experimental methods. However, approximations can be obtained by numerical methods if exact solutions are not necessary. Most of these methods, however, are neither effective nor reliable. High-precision approximation of linear and nonlinear equations is an important and challenging subject in science and engineering. As a result, several different approximation techniques have emerged, each with its own set of benefits and drawbacks. "The Adomian decomposition method, the variational iteration technique, and the Homotopy perturbation method (HPM) are among the numerous techniques that can be employed in this context. The study of variational approaches has been extensively conducted across various scientific disciplines, such as physics and engineering. The variational iteration method has demonstrated its efficacy in solving a wide range of partial differential equations (PDEs), both in cases of independence and coupling. Notable examples include the Burger's equation, coupled Burger's equations, coupled shallow water equations, linear Helmholtz PDE, Klein-Gordon PDE, and differential difference equations. Upon conducting a comparative analysis with other established methodologies such as the Adomian decomposition method, the Laplace decomposition method, and the perturbation method, it has been observed that the approach under consideration exhibits a higher level of effectiveness." The computational burden of the perturbation method becomes problematic with growing nonlinearity. However, the Adomian methodology has constraints due to the complexity of the procedures required to calculate the Adomian polynomials. A new method based on the Laplace transform that sidesteps the need for the Variational Theory was developed by Hesameddini and Latifizadeh in 2009. As a result, the new approach makes it considerably easier to calculate the Lagrange multiplier. The Homotopy Perturbation Method (HPM) is one such method that, unlike perturbation approaches, does not necessitate a vanishingly small parameter in the equations. A novel, trustworthy algorithm for recreating the variational iteration method is the goal of this study. Therefore, the Schr odinger equations will be analysed with RVIM. Nonlinear optics, plasma physics, superconductivity, and quantum mechanics all make use of Erwin Schr odinger's Schr odinger equation, which he developed to define the time-dependent quantum state of a system. For quantum mechanics, it is on par with Newton's rules for classical mechanics. According to the prevalent interpretation of quantum mechanics, the most complete description of a physical system is the quantum state, also known as a wave function or state vector. Schr odinger's equation provides a universal framework for describing systems on ranges ranging from the subatomic to the galactic. The equation was named after its creator, Erwin Schr odinger, who worked on it in 1926. The most generalised version is the Schr odinger equation, which uses time as a variable to describe a dynamic system. Both Werner Heisenberg's matrix mechanics and Richard Feynman's route integral formulation can be seen as mathematical modifications of Schr odinger's equation. The depiction of time in the Schr odinger equation causes problems for relativistic theories, however these problems are alleviated in matrix mechanics and completely disappear in the route integral formulation. There are several different methods to write the Schr odinger equation, each of which is suitable in a different physical setting. Here, we give answers to equations such;

Case 1: Linear Schr¨odinger equations,

$$u_t + iu_{xx} = 0, \quad i^2 = -1,$$

Case 2: Nonlinear Schr¨odinger equations,

$$iu_t + u_{xx} + \gamma|u|^2 u = 0, \quad i^2 = -1,$$

,

This paper investigates the utilisation of the Reconstruction of Variational Iteration Method (RVIM) for the resolution of Schrödinger equations with a specified initial condition u(x,0) = g(x), where γ and u(x,0) denote complex functions.

Basic Idea of RVIM

This section aims to provide an overview of the fundamental concepts underlying "the Reconstruction of Variational Iteration Method (RVIM) technique". We provide an overview of the RVIM method as described in references and, focusing on its application to address both linear and nonlinear Schrödinger equations. To elucidate the core principles of our proposed approach, we examine the following differential equation:

$$Lu(x_1, \ldots, x_n) + Nu(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \tag{2.1}$$

$$Lu(x_1, \ldots, x_n) = \sum_{\substack{i=0 \\ i \neq j}}^{n} L_{x_i} u, \tag{2.2}$$

Eq. (2.1) can be reformulated by considering The linear operator, denoted as L, the nonlinear operator, denoted as N, and the inhomogeneous term, denoted as f, are defined in the following manner:

$$L_{x_j} u(x_1, \ldots, x_n) = \underbrace{f(x_1, \ldots, x_n) - Nu(x_1, \ldots, x_n) - \sum_{\substack{i=0 \\ i \neq j}}^{n} L_{x_i} u}_{w(u(x_1, \ldots, x_n))}, \tag{2.3}$$

The Reconstruction of Variational Iteration Method (RVIM) is based on the underlying assumption that the equation Lxju(x1,...,xn) = w(u(x1,...,xn)) can be solved by expressing the solution u(x1,...,xn) as an infinite sum of components in the form of a series solution.

$$u(x_1, \ldots, x_n) = \lim_{n \to \infty} u_n(x_1, \ldots, x_n) = \lim_{n \to \infty} \sum_{i=0}^{n} v_i(x_1, \ldots, x_n),$$

Where 'v0' stands for the initial conditions of the main issue satisfying the equation '$L_{xj}u = 0$',"

$$v_1(x_1,\ldots,x_n) = \phi(v_0),$$
$$v_{i+1}(x_1,\ldots,x_n) = \phi\left(\sum_{k=0}^{i} v_k(x_1,\ldots,x_n)\right) - \sum_{k=1}^{i} v_k(x_1,\ldots,x_n), \quad i \geq 1,$$

$$L_{x_j}\phi(v_i) = w(v_i(x_1,\ldots,x_n)), \tag{2.4}$$

The value of ϕ(vi) is obtained with a uniform starting point. As a result, the following is obtained by applying the Laplace transform to both sides of Equation (2.4) using the standard method and utilising the initial conditions that are homogeneous.

$$P(s)\Phi_i(v_i(x_1,\ldots,x_{i-1},s,\ldots,x_n)) = \varpi(v_i(x_1,\ldots,x_{i-1},s,\ldots,x_n)), \tag{2.5}$$

The equation LAPLACE(ϕ(vi)) = Φi represents the Laplace transform of the function ϕ(vi) equal to Φi. Additionally, P(s) denotes a polynomial with a degree equivalent to the highest derivative in Equation (2.5), which is the same as the highest order of the linear operator L_{xj}. In order to

$$\mathbf{LAPLACE}(w) = \varpi, \quad \mathbf{LAPLACE}(\psi(x_i)) = \Psi(s), \quad \Psi(s) = \frac{1}{P(s)}, \tag{2.6}$$

The request is to rephrase Equation (2.5) by utilising the abbreviations "respectively" for the functions ϖ(vi(x1,...,xi−1,s,...,xn)) and w(vi(x1,...,xi−1,xi,...,xn)) in Equation (2.6). The information provided by the user is insufficient, lacking the inclusion of equations (2.5) and (2.6), as well as the necessary contextual details pertaining to these equations.

$$\Phi_i(v_i(x_1,\ldots,x_{i-1},s,\ldots,x_n)) = \varpi(v_i(x_1,\ldots,x_{i-1},s,\ldots,x_n))\Psi(s), \tag{2.7}$$

In this case, we use the convolution theorem and the inverse Laplace transform to arrive at;

$$\phi(v_i) = \int_0^{x_i} w(v_i(x_1,\ldots,x_{i-1},\tau,x_{i+1},\ldots,x_n))\psi(x_i-\tau)\,d\tau, \tag{2.8}$$

$$u_{n+1}(x_1,\ldots,x_n) = \sum_{i=0}^{n+1} v_i(x_1,\ldots,x_n) = u_0(x_1,\ldots,x_n) \tag{2.9}$$
$$+ \int_0^{x_i} w(u_i(x_1,\ldots,x_{i-1},\tau,x_{i+1},\ldots,x_n))\psi(x_i-\tau)\,d\tau,$$

It is possible to assess the whole approximation of the iteration formula by first identifying the initial approximation of u_0, from which the further approximations u_n,n > 0 can be calculated. Therefore, the precise answer can be found by using:

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{i=0}^{n} v_i, \tag{2.10}$$

Integration of the RVIM Method into the Schr¨odinger Equations

Example 3.1. The linear Schr ödinger equation is taken into consideration.

$$u_t + iu_{xx} = 0, \tag{3.1}$$

To solve Equation (3.1) using the RVIM method, we begin by specifying the initial conditions as follows: $u(x, 0) = 1 + 2cosh(2x)$. For this purpose, we choose an auxiliary linear operator denoted as $L_{tu}(x, t) = {}_{ut}$. When applying this operator to Equation (3.1), we can express the resulting equation in operator form as follows:

$$L_t u(x,t) = u_t = \underbrace{-iu_{xx}}_{w(u(x,t))}, \tag{3.2}$$

therefore $\phi(v_i)$ is defined as

$$\phi(v_i) = \int_0^{x_i} w(v_i(x,\tau))\psi(x_i - \tau)\, d\tau, \tag{3.3}$$

Using Eq.(3.3), we can easily derive the RVIM technique formulas in the t-direction needed to compute an analytical approximation to the solution of Eq. (3.1).

$$u_{n+1}(x,t) = \sum_{i=0}^{n+1} v_i(x,t) = u_0(x,t) - i\int_0^t [u_n(x,\tau)]_{xx}\, d\tau, \tag{3.4}$$

whereas, the initial approximation must be satisfy the following equation

$$\begin{aligned} &L_t u(x,t) = 0, \\ &u(x,0) = 1 + 2cosh(2x), \end{aligned}$$

therefore we begin with $u_0(x,t) = 1 + 2cosh(2x)$ accordingly by the equation (3.1) one can get the higher order approximation of the exact solution as the following relations;

$$\begin{cases} u_1(x,t) = v_0(x,t) + v_1(x,t) = 1 + 2cosh(2x)[1 - 4it], \\ u_2(x,t) = \sum_{i=0}^{2} v_i(x,t) = 1 + 2cosh(2x)[1 - 4it + \frac{(4it)^2}{2!}] \end{cases} \tag{3.5}$$

The closed-form expression for the solution is as where $u_2(x,t), n > 2$ are the remaining approximations, and each term is calculated by using the previous term.

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = 1 + 2e^{-4it}cosh(2x). \tag{3.6}$$

which is the unique solution, and it's clear that the solution obtained by RVIM method is identical to the solution obtained by using the variational iteration method, and also identical to the results obtained by homotopy perturbation method and Adomian decomposition method.

Example 3.2. we consider the linear Schr¨odinger equation

$$u_t + iu_{xx} = 0, \tag{3.7}$$

with initial condition $u(x,0) = e^{(3ix)}$, first of all, auxiliary linear operator is selected as

$$L_t u(x,t) = u_t = \underbrace{-iu_{xx}}_{w(u(x,t))},$$

Therefore by using the Eq.(3.3) we have the following RVIM's iteration formula:

$$u_{n+1}(x,t) = \sum_{i=0}^{n+1} v_i(x,t) = u_0(x,t) - i\int_0^t [u_n(x,\tau)]_{xx}\, d\tau, \tag{3.8}$$

Accordingly by using the Eq.(3.8) By making a good guess in the beginning, we can obtain a close approximation to the true solution $u_0(x,0) = v_0(x,0) = e^{(3ix)}$ as follows:

$$, \quad (3.9) \quad \begin{cases} u_1(x,t) = v_0(x,t) + v_1(x,t) = e^{(3ix)}[1+9it] \\ u_2(x,t) = \sum_{i=0}^{2} v_i(x,t) = e^{(3ix)}[1+\frac{(9it)^2}{2!}] \end{cases}$$

and so on. In the same manner, the rest of components of the iteration Eq.(3.8) can be obtained.

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^{3ix}[1+9it+\frac{(9it)^2}{2!}+\frac{(9it)^3}{3!}+\ldots]. \tag{3.10}$$

Therefore, the closed form solution is $u(x,t) = e^{3i(x+3t)}$.

It is important to note that the RVIM solution is identical to the solutions produced using the variational iteration method in, the homotopy perturbation technique in, and the Adomian decomposition method in .

Example 3.3. we consider the non-linear Schr¨odinger equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{3.11}$$

Subject to the initial condition $u(x,0) = e^{(ix)}$.

It is important to note that the RVIM solution is identical to the solutions produced using the variational iteration method in , the homotopy perturbation technique in, and the Adomian decomposition method in.

$$L_t u(x,t) = u_t = \underbrace{-u_{xx} - 2|u|^2 u}_{w(u(x,\tau))},$$

By using this assumption that u_0 is the solution of $L_t u = 0$, with initial conditions, we start with the initial approximation $u_0(x,t) = v_0(x,t) = e^{(ix)}$, where $\phi(v_i)$ is defined as

$$\phi(v_i) = -i\int_0^{x_i} w(v_i(x,\tau))\, d\tau, \tag{3.12}$$

therefore by using the Eq.(3.12) RVIM's iteration formulae in t-direction can be readily obtained as follows

$$u_{n+1}(x,t) = \sum_{i=0}^{n+1} v_i(x,t) = u_0(x,t) + i\int_0^t \left[[u_n(x,\tau)]_{xx} + 2|u_n(x,\tau)|^2 u_n(x,\tau)\right] d\tau, \tag{3.13}$$

which the subscript n indicates the nth approximation of the solution, so we can obtain the successive approximations with starting from the selected initial approximation $u_0(x,t) = e^{(ix)}$ as follows;

$$\begin{cases} u_1(x,t) = v_0(x,t) + v_1(x,t) = e^{(ix)}[1+it], \\ u_2(x,t) = \sum_{i=0}^{2} v_i(x,t) = e^{(ix)}[1+it+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}] \end{cases} \tag{3.14}$$

,

and using the fact that

$$u_{n+1}(x,t) = \sum_{i=0}^{n+1} v_i(x,t)$$

that leads to the exact solution $u(x,t) = e^{i(t+x)}$.

Notably, the RVIM method makes working with non-linear Schr odinger equations much simpler compared with other well-known approaches like the homotopy perturbation technique , the Adomian decomposition method, or the variational iteration method. The RVIM method is a powerful and convenient mathematical tool for solving nonlinear equations, and the resulting solutions agree well with the actual values.

Example 3.4. we consider the non-linear Schr¨odinger equation

$$iu_t + u_{xx} - 2|u|^2 u = 0, \tag{3.15}$$

Subject to the initial condition $u(x,0) = e^{(ix)}$. By choosing the auxiliary linear operator as

$$L_t u(x,t) = u_t = \underbrace{-u_{xx} + 2|u|^2 u}_{w(u(x,\tau))} \tag{3.16}$$

thus $\phi(v_i)$ is defined as

$$\phi(v_i) = i \int_0^t \Big[[v_i(x,\tau)]_{xx} - 2|v_i(x,\tau)|^2 v_i(x,\tau) \Big] \, d\tau, \tag{3.17}$$

so by using the Eq.(3.16) and (3.17) we obtain the following RVIM's iteration formulation in t-direction:

$$u_{n+1}(x,t) = \sum_{i=0}^{n+1} v_i(x,t) = u_0(x,t) + i \int_0^t \Big[[u_n(x,\tau)]_{xx} - 2|u_n(x,\tau)|^2 u_n(x,\tau) \Big] \, d\tau, \tag{3.18}$$

Which that The nth approximation is indicated by the subscript, given the specified initial values., we can select $u_0(x,t) = v_0(x,t) = e^{(ix)}$ and by substituting this selection in (3.18) yields the following successive approximations:

$$(3.19)\quad \begin{cases} u_1(x,t) = v_0(x,t) + v_1(x,t) = e^{(ix)}[1-3it], \\ u_2(x,t) = \sum_{i=0}^{2} v_i(x,t) = e^{(ix)}[1-3it+\frac{(3it)^2}{2!}-\frac{(3it)^3}{3!}+\frac{(3it)^4}{4!}], \\ u_n(x,t) = \sum_{i=0}^{n} v_i(x,t) = e^{(ix)}[1-3it+\frac{(3it)^2}{2!}+\ldots+(-1)^n\frac{(3it)^n}{n!}+\ldots] \end{cases}$$

,

And so on. In the same manner, the rest of components of the iteration formula, Eq.(3.18) can be obtained. Therefore, the solution of in closed form is $u(x,t) = e^{i(x-3t)}$, which that is the exact solution. The results in (3.19) are clearly the same as those of , even though our proposed iteration algorithm is much simpler.

Conclusion

In order to demonstrate the efficacy and notable features of the RVIM approach, it has been successfully applied to solving the linear and nonlinear Schr odinger equations in this study. The iteration relation of the RVIM allows for quickly convergent successive approximations to be obtained without making any limiting assumptions or undergoing any transformations that could alter the underlying physical behaviour of the issues. Since the RVIM doesn't call for the time-consuming Adomian polynomials, the iteration is simple and straightforward, and the size of the calculations is reduced as a result. The RVIM approach yields exact solutions that can be represented in a closed form. RVIM simplifies the computing effort and provides the solution quickly in comparison to the Adomian approach for nonlinear equations that emerge frequently in representing nonlinear processes. The results presented here add to the growing body of evidence supporting RVIM's ability to solve linear and nonlinear Schr odinger equations analytically and numerically.

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