WAVE PROPAGATION BETWEEN A THERMOELASTIC HALF-SPACE AND A THERMOELASTIC HALF-SPACE WITH VACUOUS PORES

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Abstract

In this paper, we have analyzed the reflection and refraction phenomena due to an incident coupled dilatational wave striking obliquely at the plane interface between a thermo-viscoelastic half space and a thermo-viscoelastic half-space with voids. Amplitude ratios of different reflected and refracted (or transmitted) waves have been provided in closed form for incidence of a set of coupled dilatational waves and basic governing equations are formulated in the framework of the three-phase-lag thermoelasticity theory. For a magnesium crystal material, expressions for reflection/refraction coefficients are computed. Through a variety of graphs, variations in the modulus values of the reflection/refraction coefficients have been shown as functions of the angle of incidence of the striking wave. These graphical representations highlight the effects of phase lag parameters and voids on the amplitude ratios of different reflected/refracted waves.

Paper Identification

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Nomenclature

τ_{ij}	Components of stress tensor
λ_e, μ_e	Lame's constants
α_0, α_1	Viscoelastic relaxation times
β_e	$(3\lambda_e + 2\mu_e)\alpha_t$
α_t	Coefficient of linear thermal expansion
c_e	Specific heat at constant strain
K	Thermal conductivity
K^*	$\frac{c_e(\lambda_e + 2\mu_e)}{4}$, material constant
T	Absolute temperature
T_0	Reference temperature
Θ	Temperature deviation from the reference temperature $\Theta = T - T_0, \left\lvert\frac{\Theta}{T_0}\right\rvert = 1$
u_i	Components of displacement vector
ρ	Density of the medium
e_{ij}	Components of strain tensor
e_{kk}	e, cubical dilatation
ϕ	Change in volume fraction field
δ_{ij}	Kronecker delta function
h_i	Components of equilibrated stress vector
q_i	Components of heat flux vector
α, b, ξ_1	Void material parameters
m	Thermo-void coefficient
χ	Equilibrated inertia
t	Time variable

1. Introduction

According to the traditional thermoelasticity concept, when an elastic material is exposed to a thermal disturbance, the effect is immediately felt at a location far from the source. This demonstrates that thermal signal vibrates at an infinitely fast rate, which is an impractical conclusion. Lord and Shulman [1] developed a generalized thermoelasticity theory with one relaxation time (single-phase-lag theory) to address this important problem. By including temperature-rate among the constitutive relations (thus creating temperature-rate-dependent thermoelasticity), Green and Lindsay [2] proposed a significant theory of generalized thermoelasticity with two relaxation times. Additionally, Tzou [3] and Chandrasekharaiah [4] offered a dual-phase-lag (DPL) heat conduction model to include the impact of microscopic interactions in the fast-transient process of heat transfer mechanism in a macroscopic formulation. According to this model, the classical Fourier's law $q_i = -K\nabla\Theta$ has been replaced by

$$q_i\left(x_i, t+\tau_q\right) = -K\nabla\Theta\left(x_i, t+\tau_T\right),$$

where the heat flux q_i generated at position x_i at time $t+\tau_q$ is the result of the temperature gradient developed at another time $t+\tau_T$ across a solid. The delay time τ_T emphasizes the micro-structural interactions. The other delay time τ_q highlights the fast-transient influences of thermal inertia. Abo-Dahab et al. [5] examined the effects of gravitational field and rotation on an electro-magneto-elastic material with diffusion and voids using the dual-phase-lag model of generalized thermoelasticity. In accordance with three different generalized thermoelasticity theories, Mondal and Othman [6] investigated the effects of memory dependent derivative on piezo-thermoelastic material using the normal mode approach.

A continuum body with voids is one that has small pores distributed uniformly throughout its volume. The traditional theory of elasticity has been extended by Cowin and Nunziato's [7] idea of elastic material with voids. According to this idea, the voids that exist in the medium are empty pores that have some surface area and volume. The bulk density of the material is expressed as the product of the matrix density and the void volume fraction field, according to the linear theory of elastic material with voids. According to the idea, the variation in the void volume fraction counts as a separate extra kinematic variable. Corresponding to this new kinematic variable, a force termed as 'equilibrated stress' is introduced, which denotes the resultant force in the matrix acting on a void as a result of its interaction with neighbouring voids. Cowin and Nunziato [7] have obtained the constitutive relations and field equations for a homogeneous elastic material with voids using the concepts of continuum mechanics. Puri and Cowin [8] were the first to investigate the possibilities of plane wave propagation in elastic material with voids and discovered that there might be three plane waves propagating at different rates.

The idea of thermoelastic solids with voids is a great generalization of the classical coupled thermoelastic model by Iesan [9]. In addition to introducing the condition for acceleration wave propagation in an isotropic, homogeneous thermoelastic solid with voids, he also supplied the fundamental governing equations. Furthermore, he stated that temperature factors and voids had little impact on the shear wave's vibration. Iesan's idea [9] has been the subject of numerous insightful research projects that have been examined by academics all around the world. A research on wave propagation in a homogeneous, isotropic generalized thermoelastic half-space with voids was published by Singh [10]. In order to tackle a two-dimensional problem of a micropolar porous circular plate with a three-phase-lag model under the umbrella of two temperature generalized thermoelasticity, Kumar et al. [11] applied

Laplace and Hankel transforms. A novel nonlocal theory of generalized thermoelastic materials with voids and fractional derivative heat transmission was presented by Bachher and Sarkar [12]. Mondal et al. [13] studied the propagation of plane waves in a nonlocal thermoelastic material with voids within the setting of dual-phase-lag (DPL) model of generalized thermoelasticity. A two-dimensional problem of thermo-mechanical interactions in a functionally graded elastic material with voids and gravity was examined by Gunghas et al. [14] using the LS theory.

2. Problem formulation

Consider a linear homogeneous, isotropic, magneto-thermo-viscoelastic solid half-space with voids ($M : 0 < z < \infty$) and a thermo-viscoelastic half-space ($M' : -\infty < z < 0$) in welded contact separated by $z = 0$. Rectangular cartesian co-ordinate system has been chosen with origin at the interface $z = 0$. The *x*-axis is taken along the interface between these half-spaces and positive *z*-axis is pointing vertically downwards into the medium M. For two-dimensional motion parallel to *xz*-plane the displacement vector $(\vec{u} = (u, v, w))$ components, change in void volume fraction ϕ and temperature Θ are

$$u = u(x, z, t),\ v = 0,\ w = w(x, z, t)\ ,\ \phi = \phi(x, z, t) \text{ and } \Theta = \Theta(x, z, t).$$

The dynamical equations for Medium *M* may be written as follows:
The constitutive relation is

$$\tau_{ij} = \lambda^* \delta_{ij} u_{k,k} + \mu^* \left(u_{i,j} + u_{j,i} \right) + b\phi\delta_{ij} - \beta^* \Theta \delta_{ij} . \tag{1}$$

The parameters λ^*, μ^* and β^* are defined as

$$\lambda^* = \lambda_e \left(1 + \alpha_0 \frac{\partial}{\partial t} \right),\ \mu^* = \mu_e \left(1 + \alpha_1 \frac{\partial}{\partial t} \right),\ \beta^* = \beta_e \left(1 + \beta \frac{\partial}{\partial t} \right)$$

where $\beta_e = \left(3\lambda_e + 2\mu_e \right) \alpha_t,\ \beta = \left(3\lambda_e \alpha_0 + 2\mu_e \alpha_1 \right) \dfrac{\alpha_t}{\beta_e}$.

The balance of linear momentum in the presence of body forces F_i may be written as

$$\rho \ddot{u}_i = \tau_{ji,j} . \tag{2}$$

Again, the volume fraction field ϕ satisfies the following equation (Iesan [15])

$$\alpha \nabla^2 \phi - b\left(\nabla \cdot \vec{u} \right) - \xi_1 \phi + m\Theta = \rho \chi \ddot{\phi} . \tag{3}$$

The heat equation corresponding to generalized thermoelasticity theory with three phase lags (Roychoudhuri [7]) is

$$\left[K^* \left(1 + \tau_\upsilon \frac{\partial}{\partial t} \right) + K \frac{\partial}{\partial t} \left(1 + \tau_T \frac{\partial}{\partial t} \right) \right] \nabla^2 \Theta = \left(1 + \tau_q + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2} \right) \left(\rho c_e \ddot{\Theta} + \beta^* T_0 \ddot{e} + m T_0 \ddot{\phi} \right). \tag{4}$$

Substituting (1) into (2), one can obtain

$$\rho \frac{\partial^2 u}{\partial t^2} = \left(\lambda^* + \mu^* \right) \frac{\partial e}{\partial x} + \mu^* \nabla^2 u - \beta^* \frac{\partial \Theta}{\partial x} + b \frac{\partial \phi}{\partial x} , \tag{5}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \left(\lambda^* + \mu^* \right) \frac{\partial e}{\partial z} + \mu^* \nabla^2 w - \beta^* \frac{\partial \Theta}{\partial z} + b \frac{\partial \phi}{\partial z} . \tag{6}$$

For convenience, we will make use of the following non-dimensional quantities

$$(x', z') = \frac{\bar{\omega}}{c_1}(x, z), \quad \left(t', \alpha_0', \alpha_1', \tau_q', \tau_\upsilon', \tau_T', \beta'\right) = \bar{\omega}(t, \alpha_0, \alpha_1, \tau_q, \tau_\upsilon, \tau_T, \beta) \ ,$$

$$(u', w') = \frac{\rho\bar{\omega}c_1}{\beta_e T_0}(u, w) \ , \ \Theta' = \frac{\Theta}{T_0}, \ \phi' = \frac{\bar{\omega}^2\chi}{c_1^2}\phi \ , \left(\tau_{zx}', \tau_{zz}'\right) = \frac{1}{\beta_e T_0}\left(\tau_{zx}, \tau_{zz}\right) \ , \qquad (7)$$

where $\bar{\omega} = \frac{\rho c_e c_1^2}{K}$, $c_1 = \sqrt{\frac{\lambda_e + 2\mu_e}{\rho}}$ are the characteristic frequency and longitudinal wave velocity in the medium respectively.

For investigation of plane waves, the potentials $\psi_1(x, z, t), \psi_2(x, z, t)$ are introduced. They are related to displacement components *u* and *w* by the relation

$$u = \frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_2}{\partial z}, \quad w = \frac{\partial\psi_1}{\partial z} + \frac{\partial\psi_2}{\partial x} \ . \qquad (8)$$

Plugging the non-dimensional quantities and potentials into Eqs.(3)-(6) under two-dimensional formulation, one can get

$$\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\nabla^2\psi_2 = a_1\frac{\partial^2\psi_2}{\partial t^2}, \qquad (9)$$

$$\left[\frac{\lambda_e}{\mu_e}\left(1 + \alpha_0\frac{\partial}{\partial t}\right) + 2\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\right]\nabla^2\psi_1 - a_1\frac{\partial^2\psi_1}{\partial t^2} - \left(1 + \beta\frac{\partial}{\partial t}\right)\gamma^2\Theta - a_2\phi = 0 \ , \qquad (10)$$

$$\nabla^2\phi - a_3\left(\nabla^2\psi_1\right) - a_4\phi + a_5\Theta - a_6\ddot{\phi} = 0 \ , \qquad (11)$$

$$\left[a_7\left(1 + \tau_\upsilon\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial t}\left(1 + \tau_T\frac{\partial}{\partial t}\right)\right]\nabla^2\Theta = \left(1 + \tau_q + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\ddot{\Theta} + a_8(1 + \beta\frac{\partial}{\partial t})\ddot{\psi}_1 + a_9\ddot{\phi}\right), \qquad (12)$$

where $a_1 = \frac{\rho c_1^2}{\mu_e}$,

$$a_2 = \frac{\rho b c_1^4}{\beta_e T_0 \mu_e \bar{\omega}^2 \chi}, \ a_3 = \frac{b\chi\beta_e T_0}{\alpha\rho c_1^2}, \ a_4 = \frac{\xi_1 c_1^2}{\bar{\omega}^2\alpha}, \ a_5 = \frac{m T_0 \chi}{\alpha},$$

$$a_6 = \frac{\rho c_1^2\chi}{\alpha}, a_7 = \frac{K^*}{K\bar{\omega}}, a_8 = \frac{\beta_e^2 T_0}{K\rho\bar{\omega}}, a_9 = \frac{m c_1^4}{K\chi\bar{\omega}^3} \ .$$

Eq. (9) is uncoupled while equations (10)-(12) are coupled in ψ_1, Θ and ϕ.

To suit the actual situation of the problem, we seek solutions of differential equations (9)-(12) in the following forms:

$$\left[\psi_1, \psi_2, \Theta, \phi\right](x, z, t) = \left[\bar{\psi}_1, \bar{\psi}_2, \bar{\Theta}, \bar{\phi}\right]\exp\{\iota k(x\sin\theta - z\cos\theta) - \iota\omega t\} \ , \qquad (13)$$

where k is the wave number and ω is angular frequency connected by the relation $\omega = kV$, V being the phase velocity and $(\sin\theta, -\cos\theta)$ denotes the projection of wave normal of incident wave onto the *xz*-plane. Barred quantities are the amplitudes of the field quantities.

Injecting Eq.(13) into Eqs.(10)-(12), we get respectively

$$\left(a_1\omega^2V^2 + \iota\omega^3 a_{10}\right)\bar{\psi}_1 + (\iota\omega V^2\gamma^2\beta_{00})\bar{\Theta} + a_2V^2\bar{\phi} = 0, \quad (14)$$

$$\omega^2 a_3\bar{\psi}_1 + a_5V^2\bar{\Theta} + (a_6\omega^2V^2 - \omega^2 - a_4V^2)\bar{\phi} = 0, \quad (15)$$

and $$\iota\omega^3\beta_{00}a_8a_{11}\bar{\psi}_1 + (\omega^2\tau_{T0} + \iota\omega a_7\tau_{\upsilon 0} + V^2a_{11})\bar{\Theta} + a_9a_{11}V^2\bar{\phi} = 0, \quad (16)$$

where $$\alpha_{00} = \alpha_0 + \frac{\iota}{\omega}, \quad \alpha_{10} = \alpha_1 + \frac{\iota}{\omega}, \quad \beta_{00} = \beta + \frac{\iota}{\omega},$$

$$\tau_{\upsilon 0} = \tau_\upsilon + \frac{\iota}{\omega}, \quad \tau_{T0} = \tau_T + \frac{\iota}{\omega},$$

$$a_{10} = \left(\frac{\lambda_e\alpha_{00} + 2\mu_e\alpha_{10}}{\mu_e}\right), \; a_{11} = 1 - \iota\omega\tau_q - \frac{\tau_q^2\omega^2}{2}.$$

The condition for the existence of non-trivial solution of the system of equations (14)-(16) provides us

$$V^6 + AV^4 + BV^2 + C = 0, \quad (17)$$

where $A = \frac{A'}{F}, B = \frac{B'}{F}, C = \frac{C'}{F}$,

$$F = a_1a_6a_{11}\omega^4 - a_1a_4a_{11}\omega^2 - a_1a_5a_9a_{11}\omega^2,$$

$$A' = a_1a_6\tau_{T0}\omega^6 + (\iota a_1a_6a_7\tau_{\upsilon 0} + \iota a_6a_{10}a_{11})\omega^5 - (a_1a_4\tau_{T0} + a_1a_{11})\omega^4$$
$$-(\iota a_1a_4a_7\tau_{\upsilon 0} + \iota a_4a_{10}a_{11} - \iota a_3a_9a_{11}\beta_{00}\gamma^2 + \iota a_5a_9a_{10}a_{11})\omega^3 - a_2a_3a_{11}\omega^2,$$

$$B' = \iota a_6a_{10}\tau_{T0}\omega^7 - (a_1\tau_{T0} + a_6a_7a_{10}\tau_{\upsilon 0})\omega^6 - (\iota a_4a_{10}\tau_{T0} + \iota a_1a_7\tau_{\upsilon 0} + \iota a_{10}a_{11})\omega^5$$
$$-(a_2a_3\tau_{T0} - a_4a_7a_{10}\tau_{\upsilon 0})\omega^4 - \iota a_2a_3a_7\tau_{\upsilon 0}\omega^3,$$

$$C' = a_7a_{10}\tau_{\upsilon 0}\omega^6 + (\iota a_8a_{11}\beta_{00} - \iota a_{10}\tau_{T0})\omega^3.$$

$V_{1,2,3}$ are the speeds of propagation of three coupled dilatational waves namely longitudinal displacement wave (P_1), thermal wave (P_2) and longitudinal void volume fraction wave (P_3). It can be easily observed that speeds of all the coupled longitudinal waves are influenced by three phase lags (τ_q, τ_υ and τ_T), viscosity and void parameters. Eq. (9) corresponds to the uncoupled transverse displacement wave (SV) whose velocity is given by

$$V_4 = \sqrt{\frac{-\iota\omega\alpha_{10}}{a_1}}. \quad (18)$$

Clearly, V_4 depends on the viscous parameters and magnetic field but is independent of thermal, void parameters and phase lags.

The field equations for thermo-viscoelastic medium M' for two-dimensional wave propagation in *xz*-plane are given by

$$\rho' \frac{\partial^2 u'}{\partial t^2} = \left(\lambda^{*\prime} + \mu^{*\prime}\right)\frac{\partial e'}{\partial x} + \mu^{*\prime}\nabla^2 u' - \gamma^{*\prime}\frac{\partial \Theta'}{\partial x}, \quad (19)$$

$$\rho' \frac{\partial^2 w'}{\partial t^2} = \left(\lambda^{*\prime} + \mu^{*\prime}\right)\frac{\partial e'}{\partial z} + \mu^{*\prime}\nabla^2 w' - \gamma^{*\prime}\frac{\partial \Theta'}{\partial z}, \quad (20)$$

$$\left[K^{*\prime}\left(1+\tau'_\upsilon \frac{\partial}{\partial t}\right) + k'\frac{\partial}{\partial t}\left(1+\tau'_T\frac{\partial}{\partial t}\right)\right]\nabla^2\Theta' = \left(1+\tau'_q\frac{\partial}{\partial t} + \frac{\tau_q'^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho' c'_e \ddot{\Theta}' + \gamma^{*\prime} T_0 \ddot{e}'\right), \quad (21)$$

where all the dashed quantities correspond to the medium M' and are having similar meanings as defined for medium M.

We will make use of following non-dimensional variables

$$(x'', z'') = \frac{\bar{\omega}'}{c'_1}(x', z'), \quad \left(t'', \alpha''_0, \alpha''_1, \tau''_q, \tau''_\upsilon, \tau''_T, \gamma''\right) = \bar{\omega}'(t', \alpha'_0, \alpha'_1, \tau'_q, \tau'_\upsilon, \tau'_T, \gamma'),$$

$$(u'', w'') = \frac{\rho'\bar{\omega}' c'_1}{\gamma'_e T_0}(u', w'), \ \Theta'' = \frac{\Theta'}{T_0}, \ \phi'' = \frac{\bar{\omega}'^2 \chi}{c_1'^2}\phi', \ (\sigma''_{zx}, \sigma''_{zz}) = \frac{1}{\gamma'_e T_0}(\sigma'_{zx}, \sigma'_{zz}), \quad (22)$$

Displacement vector components u' and w' in terms of potentials ψ'_1 and ψ'_2 are given by

$$u' = \frac{\partial \psi'_1}{\partial x} - \frac{\partial \psi'_2}{\partial z}, \quad w' = \frac{\partial \psi'_1}{\partial z} + \frac{\partial \psi'_2}{\partial x}. \quad (23)$$

Now in terms of dimensionless quantities given in (22), the equations (19)-(21) after inserting the potentials ψ'_1 and ψ'_2 along with some simplifications take the form (after dropping single primes)

$$\left(1+\alpha'_1\frac{\partial}{\partial t}\right)\nabla^2\psi'_2 = \gamma_1'^2\frac{\partial^2\psi'_2}{\partial t^2}, \quad (24)$$

$$\left[\frac{\lambda'_e}{\mu'_e}\left(1+\alpha'_0\frac{\partial}{\partial t}\right) + 2\left(1+\alpha'_1\frac{\partial}{\partial t}\right)\right]\nabla^2\psi'_1 - \gamma_1'^2\frac{\partial^2\psi'_1}{\partial t^2} - \left(1+\gamma'\frac{\partial}{\partial t}\right)\gamma_1'^2\Theta' = 0, \quad (25)$$

$$\left[a'_7\left(1+\tau'_\upsilon\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial t}\left(1+\tau'_T\frac{\partial}{\partial t}\right)\right]\nabla^2\Theta' = \left(1+\tau'_q\frac{\partial}{\partial t} + \frac{\tau_q'^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\ddot{\Theta}' + a'_8(1+\gamma'\frac{\partial}{\partial t})\nabla^2\ddot{\psi}'_1\right), \quad (26)$$

where, all the unknowns γ'_1, a'_7 and a'_8 are having similar expressions as defined for medium M with appropriate dashes.

Now, assuming the solution of the form:

$$\left[\psi'_1, \psi'_2, \Theta'\right](x, z, t) = \left[\bar{\psi}'_1, \bar{\psi}'_2, \bar{\Theta}'\right]\exp\{\iota k'_0(x\sin\theta' - z\cos\theta') - \iota\omega' t\}, \quad (27)$$

where k'_0 is the wave number and ω' is angular frequency connected by the relation $\omega' = k'_0 V'$, V' being the phase velocity. Barred quantities are the amplitudes of the field quantities.

Injecting the solutions (27) into equations (25) and (26), we get respectively

$$\left(\gamma'^2\omega'^2V'^2+\iota\omega'^3a'_{10}\right)\bar{\psi}'_1+(\iota\omega'V'^2\gamma'^2\beta'_{00})\bar{\Theta}'=0, \qquad (28)$$

and $\iota\omega'^3\beta'_{00}a'_8a'_{11}\bar{\psi}'_1+(\omega'^2\tau'_{T0}+\iota\omega'a'_7\tau'_{\upsilon0}+V'^2a'_{11})\bar{\Theta}=0$, (29)

where, the unknown quantities $\alpha'_{00},\alpha'_{10},\beta'_{00},\tau'_{\nu0},\tau'_{T0},a'_7,a'_8,a'_{10},a'_{11}$ are having similar expressions as defined for the medium M with appropriate dashes.

The condition for existence of non-trivial solution of above two equations (28) and (29), provides us following quadratic equation in V'^2

$$(V'^2)^2+A'V'^2+B'=0\ , \qquad (30)$$

where, $A'=\dfrac{A'''}{F'},\ B'=\dfrac{B'''}{F'},\ A'''=\iota\omega'^3(a'_{10}a'_{11}+\gamma'^2a'_7\tau'_{\nu0})+\omega'^4\gamma'^2(\tau'_{T0}+a'_8a'_{11}\beta'^2_{00})$,

$$B'''=\iota a'_{10}\tau'_{T0}\omega'^5-a'_7a'_{10}\tau'_{\nu0}\omega'^4,\quad F'=\gamma'^2\omega'^2a'_{11}.$$

Equation (30) is quadratic in V'^2, which implies that there shall be two dilatational waves travelling with different velocities $V'^2_{1,2}$ given by

$$V'^2_{1,2}=\frac{-A'\pm\sqrt{A'^2-4B'}}{2}. \qquad (31)$$

Equation (24) corresponds to the uncoupled transverse displacement wave (*SV*) whose velocity V'_4 is obtained by using plane wave solution (27) in equation (24) and is given by

$$V'_4=\sqrt{\frac{-\iota\omega'\alpha'_{10}}{\gamma'^2_1}}. \qquad (32)$$

3. Reflection and transmission phenomena

We shall consider the following two cases of incidence of a set of coupled longitudinal waves and a transverse wave at the interface $z=0$.

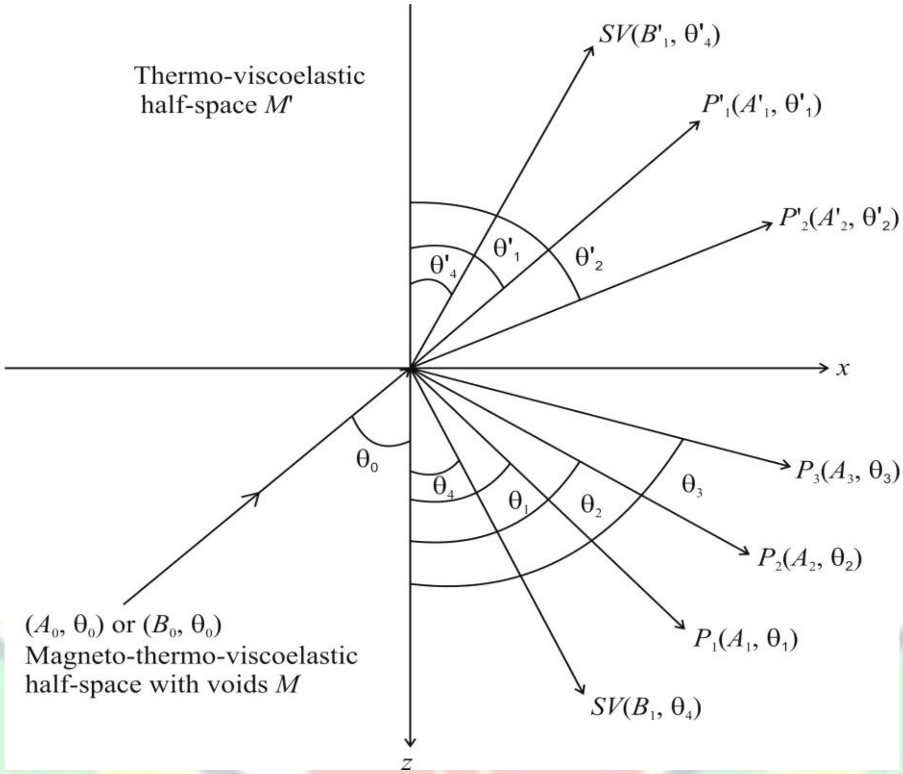

Fig. 1 Geometry of the problem (for incident P_i wave , $\theta_0=\theta_i$ (i=1,2,3))

Incidence of a coupled longitudinal wave

We assume that a set of coupled longitudinal waves of amplitude A_0 propagating with the phase velocity V_1 becomes incident obliquely at the interface, making an angle θ_0 with the normal. In order to satisfy the boundary conditions, we postulate that this incident wave gives rise to:

(1) *Reflected waves in the half-space* M :

(a) Three sets of coupled longitudinal waves with amplitudes A_1, A_2 and A_3 propagating with speeds $V_{1,2,3}$ and making angles $\theta_{1,2,3}$ respectively with the normal.

(b) A transverse wave of amplitude B_1 propagating with speed V_4 making an angle θ_4 with the normal.

(2) *Refracted waves in the half-space* M' :

(a) Two sets of coupled longitudinal waves with amplitudes A_1' and A_2' propagating with speeds $V_{1,2}'$ and making angles $\theta_{1,2}'$ respectively with the normal.

(b) A transverse wave of amplitude B_1' propagating with speed V_4' making an angle θ_4' with the normal.

In the lower medium M , full structure of the wave field consisting of the incident and reflected waves can be written as:

$$\psi_1 = A_0 \exp\{\imath k_1(x\sin\theta_0 - z\cos\theta_0) - \imath\omega_1 t\}$$

$$+\sum_{i=1}^{3} A_i \exp\{\imath k_i(x\sin\theta_i + z\cos\theta_i) - \imath\omega_i t\}, \quad (33)$$

$$\Theta = \eta_1 A_0 \exp\{\imath k_1(x\sin\theta_0 - z\cos\theta_0) - \imath\omega_1 t\}$$

$$+\sum_{i=1}^{3} \eta_i A_i \exp\{\imath k_i(x\sin\theta_i + z\cos\theta_i) - \imath\omega_i t\}, \quad (34)$$

$$\phi = \varsigma_1 A_0 \exp\{\imath k_1(x\sin\theta_0 - z\cos\theta_0) - \imath\omega_1 t\}$$

$$+\sum_{i=1}^{3} \varsigma_i A_i \exp\{\imath k_i(x\sin\theta_i + z\cos\theta_i) - \imath\omega_i t\}, \quad (35)$$

$$\psi_2 = B_1 \exp\{\imath k_4(x\sin\theta_4 + z\cos\theta_4) - \imath\omega_4 t\}. \quad (36)$$

Similarly, the full structure of the wave field of transmitted waves in medium M' may be written as

$$\psi_1' = A_1' \exp\{\imath k_1'(x\sin\theta_1' - z\cos\theta_1') - \imath\omega_1' t\}$$

$$+A_2' \exp\{\imath k_2'(x\sin\theta_2' - z\cos\theta_2') - \imath\omega_2' t\}, \quad (37)$$

$$\Theta' = \eta_1' A_1' \exp\{\imath k_1'(x\sin\theta_1' - z\cos\theta_1') - \imath\omega_1' t\}$$

$$+\eta_2' A_2' \exp\{\imath k_2'(x\sin\theta_2' - z\cos\theta_2') - \imath\omega_2' t\}, \quad (38)$$

$$\psi_2' = B_1' \exp\{\imath k_4'(x\sin\theta_4' - z\cos\theta_4') - \imath\omega_4' t\}, \quad (39)$$

where η_i, ς_i ($i = 1,2,3$) are the coupling parameters between Θ and ψ_1, ϕ and ψ_1 respectively. η_i' ($i = 1,2$) are the coupling parameters between Θ' and ψ_1'. Their expressions are given by

$$\eta_i = \frac{(a_1 a_6\omega^4 - a_1 a_4\omega^2)V_i^4 + (\imath a_6 a_{10}\omega^5 - a_1\omega^4 - \imath a_4 a_{10}\omega^3 - a_2 a_3\omega^2)V_i^2 - \imath a_{10}\omega^5}{(g_1 V_i^4 + g_2 V_i^2)}, \quad (40)$$

$$\varsigma_i = \frac{(-a_1 a_5\omega^2)V_i^4 + \imath\omega^3(a_3\beta_{00}\gamma_1^2 - a_5 a_{10})V_i^2}{(g_1 V_i^4 + g_2 V_i^2)}, \quad (41)$$

$$\eta_i' = \frac{-\left[(\gamma_1'^2\omega'^2)V_i'^2 + \imath\omega'^3 a_{10}'\right]}{\imath\omega' V_i'^2\gamma_1'^2\beta_{00}'}, \quad (42)$$

where $g_1 = \imath\omega a_4\beta_{00}\gamma_1^2 - \imath\omega^3 a_6\beta_{00}\gamma_1^2 + a_2 a_5$, $g_2 = \imath\omega^3\beta_{00}\gamma_1^2$.

The amplitudes $A_{1,2,3}$, $A_{1,2}'$, B_1 and B_1' can be determined from the boundary conditions at the interface $z = 0$. At the intersection of two distinct solid half-spaces, the following boundary conditions should be present: (i) continuity of force and stress components; (ii) continuity of displacement components; (iii) continuity of temperature; (iv) continuity of the normal heat flux component; and (v) absence of variation in the volume fraction field with distance. These boundary conditions are expressed mathematically as

$$\sigma_{zz} = \sigma_{zz}', \quad \sigma_{zx} = \sigma_{zx}', \quad u = u', \quad w = w', \quad \Theta = \Theta',$$

$$\frac{\left[k\frac{\partial}{\partial t}\left(1+\tau_T\frac{\partial}{\partial t}\right)+K^*\frac{\partial}{\partial t}\left(1+\tau_\upsilon\frac{\partial}{\partial t}\right)\right]\frac{\partial\Theta}{\partial z}}{\frac{\partial}{\partial t}\left(1+\tau_q\frac{\partial}{\partial t}\right)}=\frac{\left[k'\frac{\partial}{\partial t}\left(1+\tau'_T\frac{\partial}{\partial t}\right)+K^{*\prime}\frac{\partial}{\partial t}\left(1+\tau'_\upsilon\frac{\partial}{\partial t}\right)\right]\frac{\partial\Theta'}{\partial z}}{\frac{\partial}{\partial t}\left(1+\tau'_q\frac{\partial}{\partial t}\right)},$$

$$\frac{\partial\phi}{\partial z}=0 \qquad \text{at } z=0. \qquad (43)$$

The non-dimensional form of first four boundary conditions in terms of potentials $\psi_{1,2}$ and $\psi'_{1,2}$ can be written as

$$\gamma_e\left[\left(\delta_2\left(1+\alpha_0\frac{\partial}{\partial t}\right)\right)\nabla^2\psi_1+2\delta_3\left(1+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial z}+\frac{\partial\psi_2}{\partial x}\right)+b_1\phi\right.$$
$$\left.-\left(1+\gamma\frac{\partial}{\partial t}\right)\Theta\right]=\gamma'_e\left[\delta'_2\left(1+\alpha'_0\frac{\partial}{\partial t}\right)\nabla^2\psi'_1+2\delta'_3\left(1+\alpha'_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial z}\left(\frac{\partial\psi'_1}{\partial z}+\frac{\partial\psi'_2}{\partial x}\right)\right.$$
$$\left.-\left(1+\gamma'\frac{\partial}{\partial t}\right)\Theta'\right], \qquad (44)$$

$$\gamma_e\left[\delta_3\left(1+\alpha_1\frac{\partial}{\partial t}\right)\left(2\frac{\partial^2\psi_1}{\partial x\partial z}+\frac{\partial^2\psi_2}{\partial x^2}-\frac{\partial^2\psi_2}{\partial z^2}\right)\right]$$
$$=\gamma'_e\left[\delta'_3\left(1+\alpha'_1\frac{\partial}{\partial t}\right)\left(2\frac{\partial^2\psi'_1}{\partial x\partial z}+\frac{\partial^2\psi'_2}{\partial x^2}-\frac{\partial^2\psi'_2}{\partial z^2}\right)\right], \qquad (45)$$

$$\frac{\gamma_e}{\rho\bar{\omega}c_1}\left(\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial z}\right)=\frac{\gamma'_e}{\rho'\bar{\omega}'c'_1}\left(\frac{\partial\psi'_1}{\partial x}-\frac{\partial\psi'_2}{\partial z}\right), \quad (46)$$

$$\frac{\gamma_e}{\rho\bar{\omega}c_1}\left(\frac{\partial\psi_1}{\partial z}+\frac{\partial\psi_2}{\partial x}\right)=\frac{\gamma'_e}{\rho'\bar{\omega}'c'_1}\left(\frac{\partial\psi'_1}{\partial z}+\frac{\partial\psi'_2}{\partial x}\right), \quad (47)$$

where $\delta_2=\dfrac{\lambda_e}{\rho c_1^2},\ \delta_3=\dfrac{\mu_e}{\rho c_1^2},\ \delta'_2=\dfrac{\lambda'_e}{\rho' c_1'^2},\ \delta'_3=\dfrac{\mu'_e}{\rho' c_1'^2},\ b_1=\dfrac{bc_1^2}{\gamma_e T_0\bar{\omega}^2\chi}$.

Relation among the wave numbers $k_i,\ k'_j$ and angles θ_i,θ'_j for $i=1,2,3,4,\ j=1,2,4$ is given by Snell's law as below

$$k_1\sin\theta_1=k_2\sin\theta_2=k_3\sin\theta_3=k_4\sin\theta_4=k'_1\sin\theta'_1=k'_2\sin\theta'_2=k'_4\sin\theta'_4. \qquad (48)$$

Now, substituting the values of potentials $\psi_{1,2},\Theta,\phi,\ \psi'_{1,2}$ and Θ' from (33)-(39) into the above boundary conditions, assuming that all frequencies are same at the interface and making use of expression (48), we can obtain the following system of simultaneous equations

$$\sum A_{ij}Z_j=C_i, \qquad (i,j=1,2,3,4,5,6,7), \qquad (49)$$

where $A_{1j}=\gamma_e\left[a_{12}-\delta_4+a_{13}\cos^2\theta_j+b_1\dfrac{\varsigma_j}{k_j^2}+a_{14}\dfrac{\eta_j}{k_j^2}\right]\dfrac{k_j^2}{k_1^2}$,

$$A_{2j} = \frac{1}{2}\gamma_e a_{13} \sin 2\theta_j \frac{k_j^2}{k_1^2}, \quad A_{3j} = a_{18}\iota \sin\theta_j \frac{k_j}{k_1}, \quad A_{4j} = a_{18}\iota \cos\theta_j \frac{k_j}{k_1}$$

$$A_{5j} = \eta_j, \quad A_{6j} = \iota a_{19} \cos\theta_j \eta_j \frac{k_j}{k_1}, \quad A_{7j} = \varsigma_j \cos\theta_j \frac{k_j}{k_1} \quad \text{for} \quad (j = 1, 2, 3),$$

$$A_{14} = \gamma_e a_{13} \sin\theta_4 \cos\theta_4 \frac{k_4^2}{k_1^2}, \quad A_{24} = -\frac{1}{2} a_{13}\gamma_e \cos 2\theta_4 \frac{k_4^2}{k_1^2}, \quad A_{34} = -\iota a_{18} \cos\theta_4 \frac{k_4}{k_1},$$

$$A_{44} = \iota a_{18} \sin\theta_4 \frac{k_4}{k_1}, \quad A_{54} = 0, \quad A_{64} = 0, \quad A_{74} = 0,$$

$$A_{1j} = -\gamma'_e \left[a_{15} + a_{16} \cos^2\theta'_{j-4} + a_{17} \frac{\eta'_{j-4}}{k'^2_{j-4}} \right] \frac{k'^2_{j-4}}{k_1^2}, \quad A_{2j} = -\gamma'_e a_{16} \sin\theta'_{j-4} \cos\theta'_{j-4} \frac{k'^2_{j-4}}{k_1^2},$$

$$A_{3j} = \iota \sin\theta'_{j-4} \frac{k'_{j-4}}{k_1}, \quad A_{4j} = \iota \cos\theta'_{j-4} \frac{k'_{j-4}}{k_1}, \quad A_{5j} = -\eta'_{j-4}, \quad A_{6j} = \iota \cos\theta'_{j-4}\eta'_{j-4} \frac{k'_{j-4}}{k_1},$$

$$A_{7j} = 0 \quad \text{for} \quad (j = 5, 6),$$

$$A_{17} = \gamma'_e a_{16} \sin\theta'_4 \cos\theta'_4 \frac{k'^2_4}{k_1^2}, \quad A_{27} = -\frac{1}{2}\gamma'_e a_{16} \cos 2\theta'_4 \frac{k'^2_4}{k_1^2}, \quad A_{37} = -\iota \cos\theta'_4 \frac{k'_4}{k_1},$$

$$A_{47} = -\iota \sin\theta'_4 \frac{k'_4}{k_1}, \quad A_{57} = 0, \quad A_{67} = 0, \quad A_{77} = 0,$$

$$a_{12} = \iota\omega\delta_2\alpha_{00}, \quad a_{13} = 2\iota\omega\delta_3\alpha_{10}, \quad a_{14} = \iota\omega\beta_{00}, \quad a_{15} = \iota\omega\delta'_2\alpha'_{00}, \quad a_{16} = 2\iota\omega\delta'_3\alpha'_{10},$$

$$a_{17} = \iota\omega\beta'_{00}, \quad a_{18} = \frac{\gamma_e \rho' \bar{\omega}' c'_1}{\gamma'_e \rho \bar{\omega} c_1}, \quad a_{19} = \frac{[-\iota\omega k'(1-\iota\omega\tau'_T) + K^{*\prime}(1-\iota\omega\tau'_\upsilon)]}{[-\iota\omega k(1-\iota\omega\tau_T) + K^*(1-\iota\omega\tau_\upsilon)]} \frac{(1-\iota\omega\tau_q)}{(1-\iota\omega\tau'_q)},$$

$$C_1 = -A_{11}, \quad C_2 = A_{21}, \quad C_3 = -A_{31}, \quad C_4 = A_{41}, \quad C_5 = -A_{51}, \quad C_6 = A_{61}, \quad C_7 = -A_{71},$$

$$Z_1 = \frac{A_1}{A_0}, \quad Z_2 = \frac{A_2}{A_0}, \quad Z_3 = \frac{A_3}{A_0}, \quad Z_4 = \frac{B_1}{A_0}, \quad Z_5 = \frac{A'_1}{A_0}, \quad Z_6 = \frac{A'_2}{A_0}, \quad Z_7 = \frac{B'_1}{A_0}.$$

Here, $Z_{1,2,3,4}$ are the reflection coefficients, while $Z_{5,6,7}$ are the refraction coefficients for the incidence of a set of coupled dilatational wave travelling with speed V_1.

4. Numerical results and discussion

We have taken into account an example where magnesium crystal material is treated as an isotropic thermo-viscoelastic solid for computations of amplitude ratios of various reflected and transmitted waves in order to analyze this subject in more detail. Thus for the half-space M, we have

$$\lambda_e = 2.17\times10^{10}\, Nm^{-2},\ \mu_e = 3.278\times10^{10}\, Nm^{-2},\ \gamma_e = 2.68\times10^{6}\, Nm^{-2}\,\text{degree}^{-1},\ T_0 = 298K,$$
$$k = 1.7\times10^{2} Wm^{-1}\,\text{degree}^{-1},\ c_e = 1.04\times10^{3}\, JKg^{-1}\,\text{degree}^{-1},\ \rho = 1.74\times10^{3}\, Kgm^{-3}.$$

For the half-space, M' :

$$\lambda'_e = 2.12\times10^{10}\,Nm^{-2},\ \mu'_e = 3.17\times10^{10}\,Nm^{-2},\ \gamma'_e = 1.07\times10^{6}\,Nm^{-2}\,\text{degree}^{-1},\ T_0 = 298K,$$
$$k' = 1.14\times10^{2}Wm^{-1}\,\text{degree}^{-1},\ c'_e = 0.5977\times10^{3}\,JKg^{-1}\,\text{degree}^{-1},\ \rho' = 3.8\times10^{3}\,Kgm^{-3}.$$

Void parameters are given by

$$\alpha = 3.688\times10^{-5}\,N,\quad \xi_1 = 1.475\times10^{10}\,Nm^{-2},\ \chi = 1.753\times10^{-15}\,m^2,$$
$$b = 1.13849\times10^{10}\,Nm^{-2},\quad m = 2\times10^{6}\,Nm^{-2}\,\text{degree}^{-1}.$$

Other constants involved in the problem are taken as:

$$\tau_\upsilon = \tau'_\upsilon = 0.1,\ \tau_q = \tau'_q = 0.2,\ \tau_T = \tau'_T = 0.15,\ \alpha_0 = \alpha'_0 = 0.06,\ \alpha_1 = \alpha'_1 = 0.09,$$
$$\omega = 45.$$

We assessed the reflection/refraction coefficients in light of the aforementioned physical information. All of the amplitude ratios are discovered to have complex values, as was anticipated beforehand. Comparisons of the reflection/refraction coefficients have been done within the context of thermo-viscoelastic theory based on:

(i) Three-phase-lag model with voids (3PLV) shown by solid line,

(ii) GN-III model with voids (GN3V) shown by dashed line,

(iii) Three-phase-lag model without voids (3PLWV) shown by dotted line.

Figures 2-8 are meant for the case of incidence of a *P*-wave propagating with speed V_1. Considered range for angle of incidence is $0^\circ \le \theta_0 \le 90^\circ$. The modulus values of the reflection coefficients are presented in figure 2 as a function of the angle of incidence. It is evident that, with the exception of the 3PLWV model, has value nearly equal to unity over the entire incidence range. The numerical values for the 3PLV and GN3V models differ little, as shown in Figure, illuminating the fact that the three phase lag factors have only a minor influence on this reflection coefficient. Moreover, presence of voids increases the values of $|Z_1|$ in the entire range of angle of incidence.

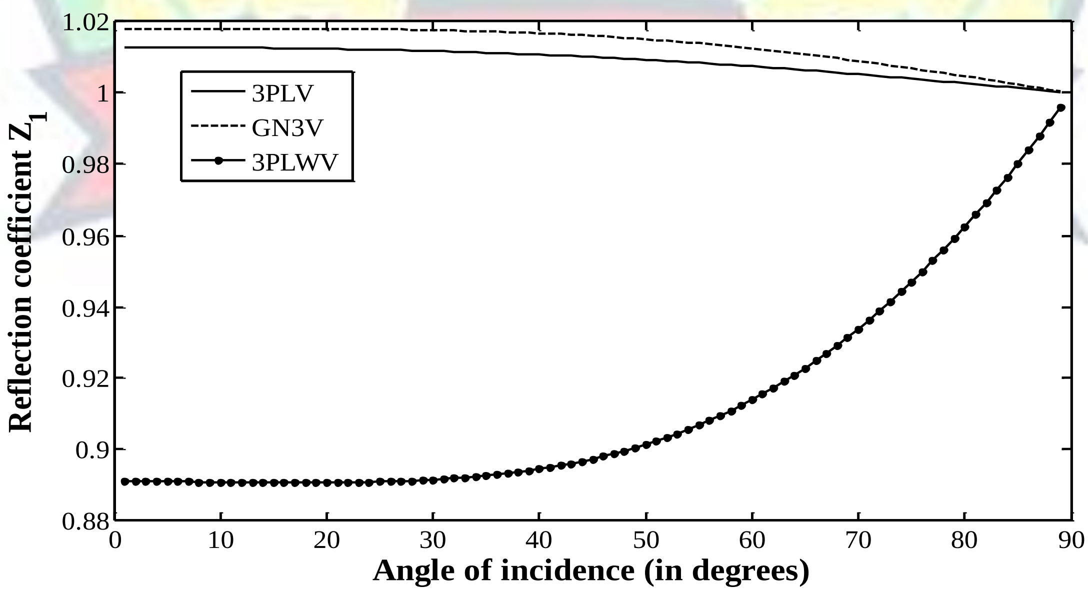

Fig. 2 Variation of the modulus of reflection coefficient Z_1 with angle of incidence of coupled longitudinal wave with speed V_1

We compared the fluctuations in the modulus values of the reflection coefficient in figure 3. Modulus values begin with a maximum value close to normal incidence, then decline with increasing incidence angle, finally becoming zero close to grazing incidence. As illustrated, variations in reflection coefficient $|Z_2|$ follow same trend for all the three models. Clearly, Z_2 is significantly affected due to void parameters and relaxation times. Presence of voids and three phase lag parameters is responsible for decrement in the values of $|Z_2|$.

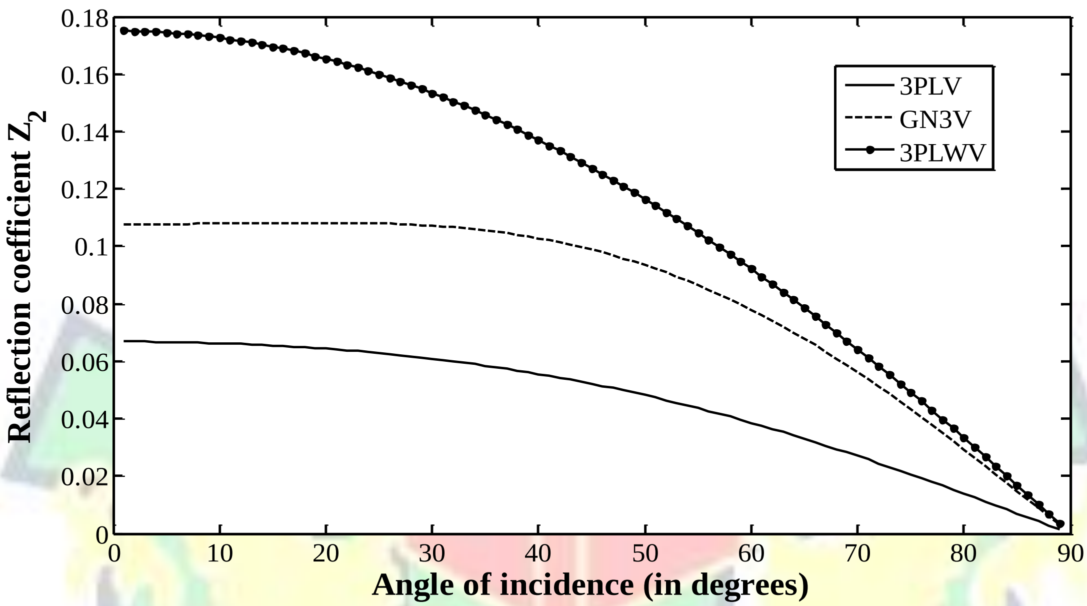

Fig. 3 Variation of the modulus of reflection coefficient Z_2 with angle of incidence of coupled longitudinal wave with speed V_1

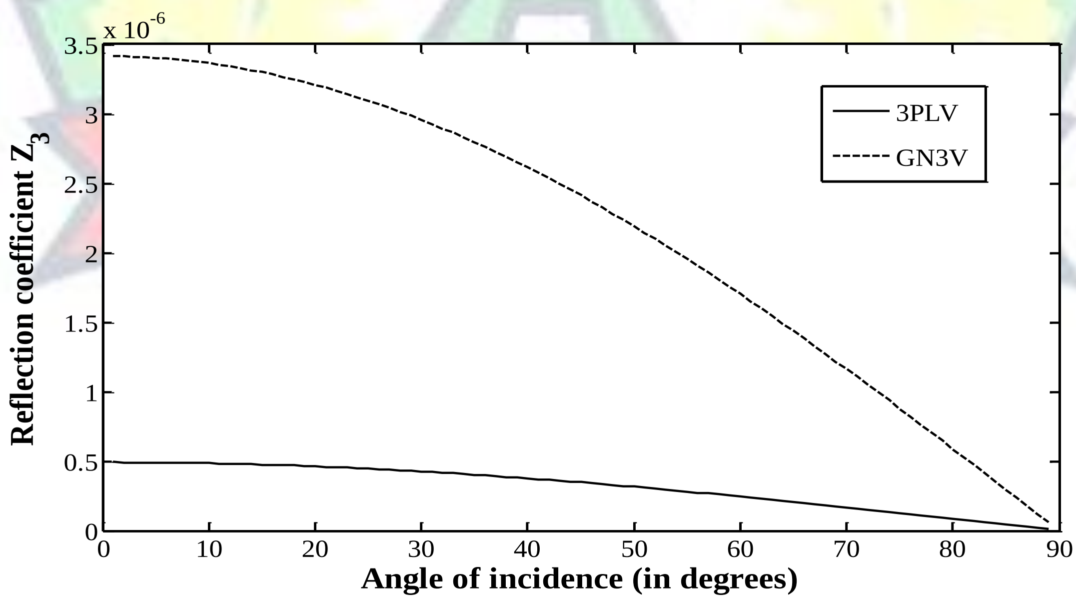

Fig. 4 Variation of the modulus of reflection coefficient Z_3 with angle of incidence of coupled longitudinal wave with speed V_1

Figure 4 is depicting a comparison of the profile of reflection coefficient $|Z_3|$ with increasing angle of incidence. Magnitude of Z_3 is very small for both the cases during the whole range of incidence. Pattern of variations for both the models 3PLV and GN3V is similar. Clearly, presence of three relaxation times τ_q, τ_υ and τ_T decreases the modulus values of Z_3.

Figure 5 is characterizing the behaviour of modulus values of Z_4 as a function of angle of incidence. The variations in Z_4 is alike for all the models with different degrees of magnitude. Magnitude of Z_4 increases from zero value at 1° angle of incidence, attains maximum value at $\theta_0 = 45^\circ$, then decreases with further increase in angle of incidence and ultimately vanishes at grazing incidence. It is noticed from the plot that magnitude of reflection coefficient Z_4 gets suppressed due to the absence of porosity and presence of phase lag parameters in the medium.

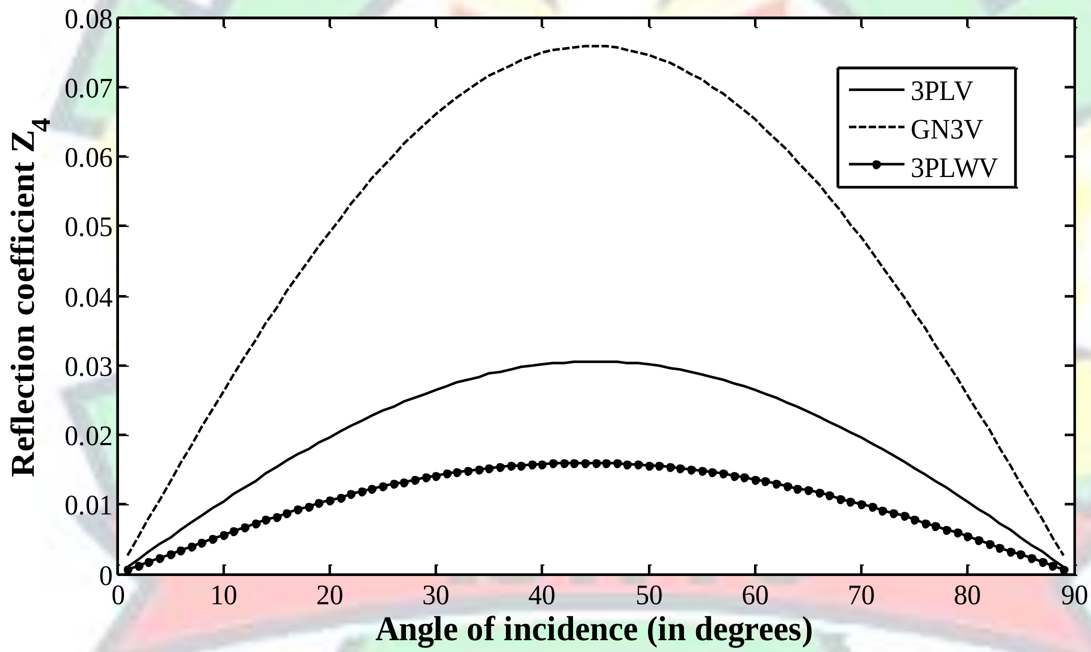

Fig. 5 Variation of the modulus of reflection coefficient Z_4 with angle of incidence of coupled longitudinal wave with speed V_1

The solution curves for the magnitude of amplitude ratio $|Z_5|$ obtained for the refracted wave with speed V_1' are portrayed through figure 6. Variations in $|Z_5|$ are similar to that for Z_2 and Z_3. It can be inferred from the figure that presence of porosity in the medium acts as a decreasing agent for $|Z_5|$. It is also worth noticing here that magnitude of Z_5 is high for GN3V model as compared to 3PLV model entailing that absence of phase lag parameters acts as an increasing agent for $|Z_5|$.

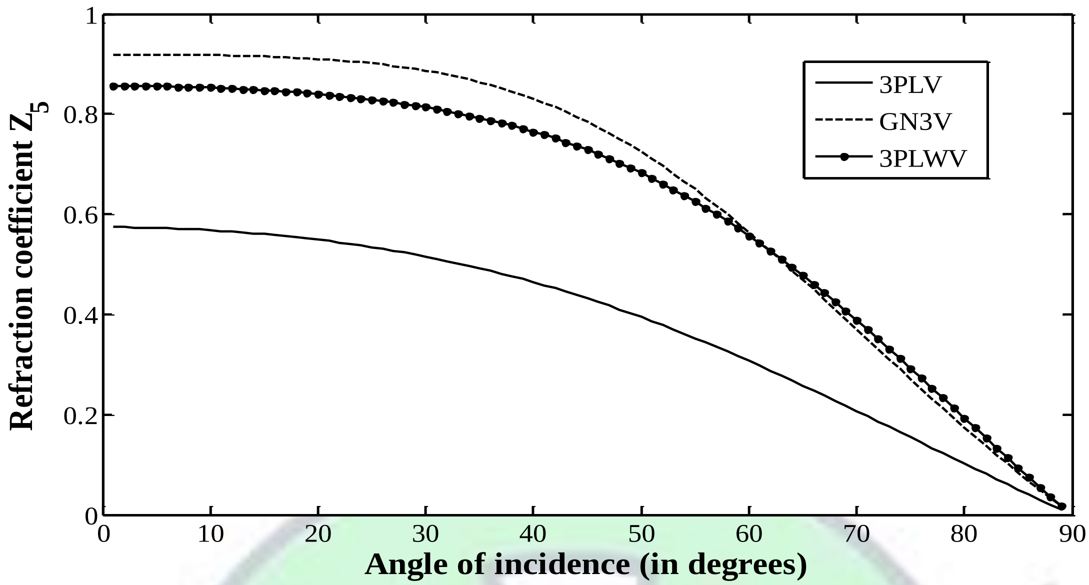

Fig. 6 Variation of the modulus of refraction coefficient Z_5 with angle of incidence of coupled longitudinal wave with speed V_1

To observe the effects of voids and relaxation times τ_q, τ_υ and τ_T on the modulus values of amplitude ratio Z_6 for the refracted wave having speed V_2', we refer to figure 7. The refraction coefficient $|Z_6|$ begins with its maximum value at 1° angle of incidence and afterwards it behaves as a monotonically decreasing function during the whole range of incidence. It vanishes near 90° angle of incidence. Consideration of voids in the medium magnifies the modulus values of Z_6 while phase lag parameters are found to decrease the values of $|Z_6|$.

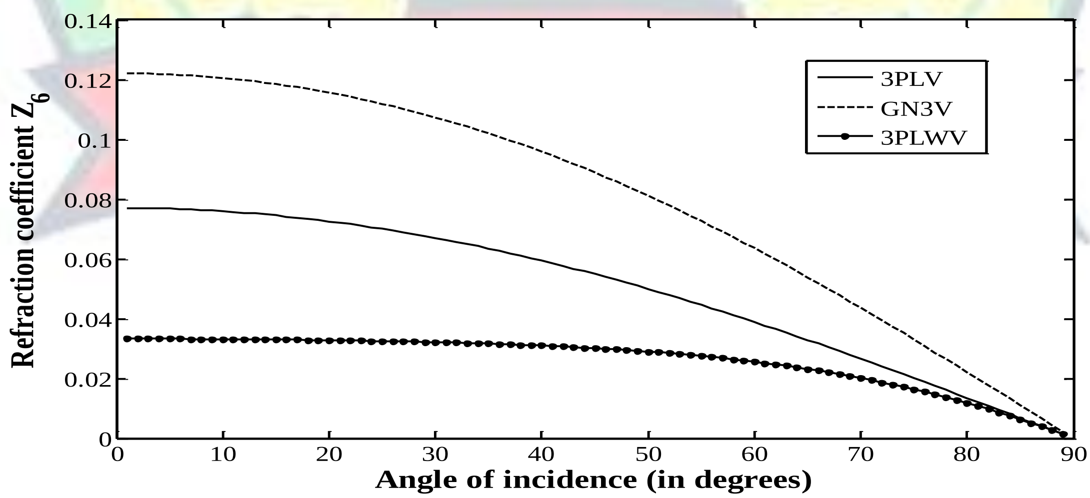

Fig. 7 Variation of the modulus of refraction coefficient Z_6 with angle of incidence of coupled longitudinal wave with speed V_1

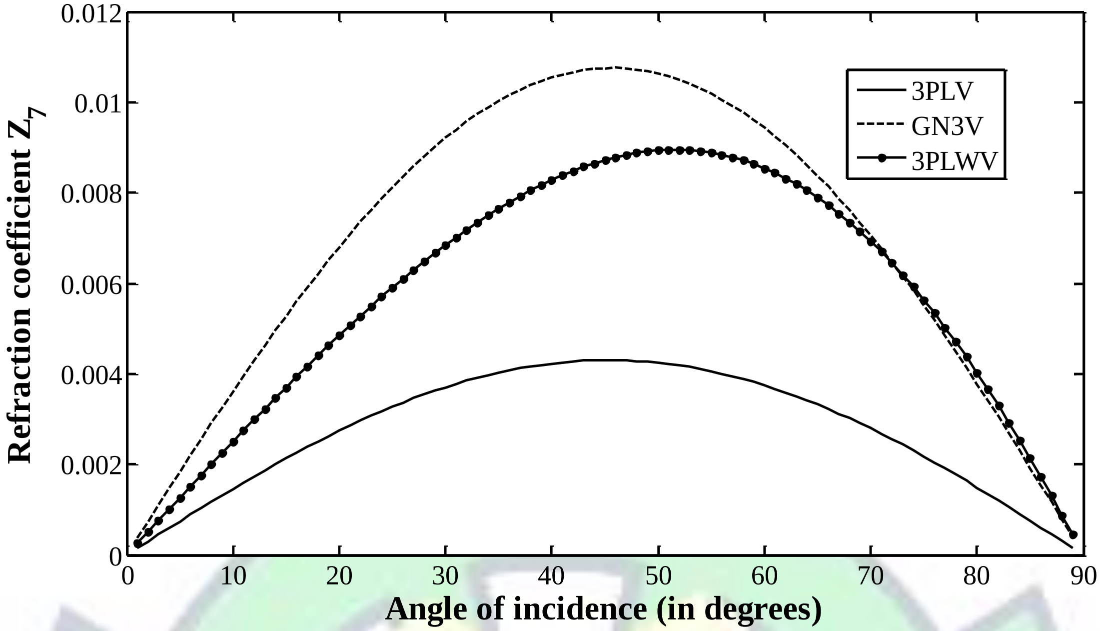

Fig. 8 Variation of the modulus of refraction coefficient Z_7 with angle of incidence of coupled longitudinal wave with speed V_1

Figure 8 is devoted to analyse the variations in the modulus values of amplitude ratio Z_7 corresponding to refracted transverse wave moving with speed V_4'. Its trend of variations resembles Z_4. Presence of void parameters and phase lag parameters causes a decrement in the values of $|Z_7|$.

5. Concluding remarks

1. The phase speeds of all the existing waves are found to be complex valued and frequency dependent.
2. At grazing incidence ($\theta = 90^\circ$) of longitudinal wave of speed V_1, no other reflected/refracted wave appears except the longitudinal wave of the same amplitude as that of incident wave. Thus no reflection/refraction takes place at grazing incidence.
3. Voids and three phase lag parameters are having a significant effect on reflection/refraction coefficients.

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