QUANTIFYING INEQUALITY: A STUDY OF MEAN-MEDIAN-MODE RELATIONSHIPS IN GROUPED DATA

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Abstract

In statistical analysis, the mean, median, and mode show what percentage of data falls in the middle. Aggregated data may alter the typical placement of the median between the mean and the mode for skewed unimodal distributions, which is between the two extremes. Because of this, it is difficult to accurately evaluate this characteristic using summary statistics alone. To address this vacuum in the statistical literature, this work analyses the mean-median-mode inequality for grouped data. The complicated relationships between the mean, median, and mode of unimodal clustered distributions are explored. We propose a thorough methodology for investigating the relationship between these fundamental indicators by establishing the conditions that underlie their differences. By examining several classified datasets, we are able to adapt our findings to real-world scenarios. Statistical approaches are used to examine and interpret the data, taking into account things like sample size and the dispersion of results. Our research shows that there are discrepancies between the mean, median, and mode that were not expected. In this research, we focus on the significance of studying mean-median-mode inequality without complete raw data, making it applicable to real-world data analysis. The findings might be useful for researchers, statisticians, and data analysts that work with grouped data. Statistics, decision-making, and predictive modelling may all benefit from a deeper familiarity with the relationships between an aggregate's mean, median, and mode. In order to fill a need in the literature and enhance understanding of central tendency measures in grouped data, our work sheds light on this often-overlooked aspect of statistical analysis.

Paper Identification

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1. Introduction

Numerous academic sources, such as textbooks, journal articles, and statistical reports, have examined the connection between the mean, median, mode, and skewness, often referred to as the mean-median mode inequality, in both continuous and discrete distributions. It is widely accepted that in right-skewed distributions, the mean is typically greater than the median, while in left-skewed distributions, the mean is usually less than the median. However, the median consistently falls between the mode and mean. Nevertheless, Von Hippel (2005) demonstrated that this rule does not apply in multimodal distributions or distributions with one long tail and one heavy tail. Additionally, it fails in discrete distributions where the areas on either side of the median are unequal. In such cases, the median can be located further out in the long tail than the mean, especially when the areas on either side of the median are unequal for discrete variables. Similar exceptions exist for continuous variables, particularly when the density function is bimodal or when one tail is long and the other is heavy. On the contrary, certain studies and researchers have provided examples where the classic mean-median-mode inequality holds. Conversely, Abadir (2005) presented counterexamples suggesting that the mean-median-mode inequality is violated when the first three moments of a random variable X exist and its density is unimodal with positive skewness. Similarly, Bottomley (2002) found that the median is not always situated between the mean and mode for a unimodal distribution. The ratio (median-mean)/(mode-mean) can be negative, zero, or positive, and the difference between the median and mean can take any real value, even when the mean and mode are identical. The $\chi2$ distribution is commonly encountered in inferential statistics, and to comprehend the mean-median-mode inequality, the ordering of the mean, median, and mode in the central $\chi2$ distribution has been extended to the noncentral case (Sen, 1989). In the noncentral case, the mean was found to be additive, while both the median and mode of the F distribution were found to be sub-additive concerning the non-centrality parameter. Thus, in this context, the mean-median-mode inequality can be summarized as $M \geq Me \geq Mo \text{ or } M \geq Mo \geq Me$. Early research on the mean-median-mode inequality was conducted by Groenveld & Meeden (1977), who developed sufficient conditions on the density functions to establish inequalities among the mean, mode, median, and skewness. Furthermore, Sato (1997) presented a comprehensive theorem that incorporated previously known results, emphasizing the extension of results or sufficient conditions in cases where a density function does not exist. For most asymmetric unimodal continuous distributions, including skewed normal distributions, skewed t-distributions, and non-central $\chi 2$ distributions, the median falls between the mean and mode. However, when analyzing grouped data where complete raw data is not always available, it is necessary to study the relationship between the mean, median, and mode, and derive the necessary and/or sufficient conditions for the mean-median-mode inequality with grouped data, as well as its application to real data."

2. Mean, median and mode of asymmetric unimodal continuous distributions

We consider some examples of unimodal asymmetric continuous distribution. For $\chi^2(k)$ distribution, the probability density function (pdf) is

$$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, x \in [0, +\infty).$$

Let $f^0(x) = 0$, then we have $M_o = \max\{k - 2, 0\}$.

We can find the median by solving the following Volterra integral equation:

$$\frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} \int_0^{M_e} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \, dx = \frac{1}{2}.$$

It is well known that the median $M_e \approx k(1 - \frac{2}{9k})^3$, the mean $M = k$, and the skewness $S_k = \sqrt{\frac{8}{k}}$ for $\chi^2(k)$ distribution (Box et al., 1978). From Table 1, one can see that median M_e is less than mean M but larger than mode M_o for $\chi^2(k)$ distributions. We have the inequality $M_o < M_e < M$ for any degrees of freedom k.

Table 1 Skewness, mean, median, and mode for $\chi 2$ (k) distributions

df: k	1	2	3	4	5	10	20	50	100	200
M	1	2	3	4	5	10	20	50	100	200
M_e	0.47	1.41	2.38	3.37	4.36	9.35	19.3	49.3	99.3	199.3
M_o	0	0	1	2	3	8	18	48	98	198
S_k	2.83	2.00	1.63	1.41	1.27	0.89	0.63	0.40	0.28	0.20

For another example, we consider the standard Weibull distributions. The probability density function of a standard Weibull random variable is given by

$$f(x) = kx^{k-1}e^{-x^k},\ x \geq 0;\ k > 0.$$

The mean $M = \Gamma(1 + \frac{1}{k})$, median $M_e = \sqrt[k]{\ln 2}$, mode $M_o = (\frac{k-1}{k})^{\frac{1}{k}}$ if $k > 1$, $M_o = 0$, if $k = 1$, variance $\sigma^2 = \Gamma(1 + \frac{2}{k}) - (\Gamma(1 + \frac{1}{k}))^2$, skewness $S_k = \frac{\Gamma(1+\frac{3}{k})-3M\sigma^2-M^3}{\sigma^3}$ (Johnson et al., 1994). From Table 2, one can see that median M_e is always between mean M and mode M_o no matter it is right skewed or left skewed. First we have the inequality $M_o < M_e < M$ when the parameter $k \leq 1$ since it is positively skewed when $k \leq 3$. Then we have the inequality $M < M_e < M_o$ when the parameter $k > 3$ since it is negatively skewed when $k > 3$.

Table 2 Normal Wellbull Distribution Skewness Mean Median Mode

df: k	1	2	3	4	5	20	100	200
M	1	0.886	0.893	0.906	0.918	0.974	0.994	0.997
M_e	0.693	0.833	0.885	0.912	0.929	0.982	0.996	0.998
M_o	0	0.707	0.874	0.931	0.956	0.997	1.000	1.000
S_k	2	0.631	0.168	-.087	-0.254	-0.868	-1.082	-1.110

3. Mean, median, and mode calculations for grouped data

We are going to assume that there are k groups, and that the total number of people in the sample is n. Each group shares a width, designated here by the letter d. are only few of the sources that include the formulae for finding the mean, median, and mode when working with grouped data. Other sources include.

The following is a definition of the formula that may be used to compute the mean using grouped data:

$$M = \frac{\sum_{i=1}^{k}(C + (i-1) \times d) \times f_i}{\sum_{i=1}^{k} f_i}, \quad (1)$$

The mean is denoted by M, the midpoint of the first group is denoted by C, and the frequency of the i^{th} group is denoted by f1. It should be obvious that (1) may be rewritten as

$$M = C + \frac{d \times \sum_{i=1}^{k}(i-1) \times f_i}{\sum_{i=1}^{k} f_i}. \quad (2)$$

The following is a definition of the formula that may be used to compute the median from grouped data:

$$M_e = L + \frac{\frac{n}{2} - F}{f_m} \times d, \quad (3)$$

where M_e represents the median, L represents the lower class limit of the group that contains the median, n represents the overall frequency (number of data), F represents the cumulative frequency of the groups that came before the median group, and f_m represents the frequency of the median group. The following is a definition of the formula that may be used to compute the mode from grouped data:

$$M_o = L + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times d, \quad (4)$$

where M_o is the mode, L is the lower limit of the modal group, Δ_1 is the difference between the frequencies in the modal group and the premodal group or $\Delta_1 = f_m - f_m - 1$, Δ_2 is the difference between the frequencies in the modal group and the postmodal group or $\Delta_2 = f_m - f_{m+1}$. Throughout this paper, we assume that the mode is unique for any given sample of grouped data.

4. Prerequisites for a set of grouped data to have mean-median-mode inequalities

If the formulae that were developed in the previous section are used for the computation, then all six of the potential mean-median-mode inequalities will hold true for the grouped data. These three measures of central tendency might vary, yet still be considered part of the same group or belong to an entirely new group.

Theorem 1. We assume that there are k groups and k is odd, the mode is within the m^{th} group, $k = 2m - 1$. The necessary and sufficient condition for which $M_o \leq M$:

$$A \geq \frac{k}{2} + \frac{\Delta_1}{\Delta_1 + \Delta_2},$$

$$\text{where } A = \frac{\sum_{i=1}^{k} i f_i}{n}. \quad (5)$$

Proof. Assume $M_o \leq M$, then we have $L + \frac{\Delta_1 \times d}{\Delta_1 + \Delta_2} \leq C + \frac{\sum_{i=1}^{k}(i-1) f_i}{n} \times d$ since

$$M_o = L + \frac{\Delta_1 \times d}{\Delta_1 + \Delta_2}$$

and

$$M = C + \frac{\sum_{i=1}^{k}(i-1) f_i}{n} \times d,$$

where C is the midpoint of the first group, then we have

$$C + \left(\frac{k+1}{2} - 1\right)d - \frac{1}{2}d + \frac{\Delta_1 d}{\Delta_1 + \Delta_2} \leq C + \frac{\sum_{i=1}^{k}(i-1) f_i d}{n}$$

$$\Longrightarrow \frac{k-2}{2} + \frac{\Delta_1}{\Delta_1 + \Delta_2} \leq \frac{\sum_{i=1}^{k}(i-1) f_i}{n}.$$

Thus, we have

$$A \geq \frac{k}{2} + \frac{\Delta_1}{\Delta_1 + \Delta_2} \text{ if } M_o \leq M.$$

On the other hand, we have $M_o \leq M$ if $A \geq \frac{k}{2} + \frac{\Delta_1}{\Delta_1+\Delta_2}$.

Note that the mode and the mean tend to belong to the same set. Yet Theorem 1 can be proven without making this assumption.

Theorem 2. We are going to make the assumption that the mean and the median are both part of the same group, which we will call the m_{th} group. "The cumulative frequency of the groups before the median or modal group is denoted by the letter F, while the cumulative frequency of the groups after the median or modal group is denoted by the letter G. The condition that is both required and sufficient for which $M_o \leq M_e$:

$$G - F \geq f_m \frac{\Delta_1 - \Delta_2}{\Delta_1 + \Delta_2}. \quad (6)$$

Proof. Assuming $M_o \leq M_e$, then we have

$$L + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times d \leq L + \frac{\frac{1}{2}n - F}{f_m} \times d$$

$$\frac{\Delta_1}{\Delta_1 + \Delta_2} \leq \frac{\frac{n}{2} - F}{f_m}.$$

Thus, $2 f_m(f_m - f_{m-1}) \leq (n - 2F)(2 f_m - f_{m+1} - f_{m-1})$,

or

$$\begin{aligned} 2 f_m \Delta_1 &\leq (n - 2F)(\Delta_1 + \Delta_2) \\ &= (F + f_m + G - 2F)(\Delta_1 + \Delta_2) \\ &= (f_m + G - F)(\Delta_1 + \Delta_2). \end{aligned}$$

Hence we have

$$G - F \geq f_m \frac{\Delta_1 - \Delta_2}{\Delta_1 + \Delta_2}.$$

Similarly, we can have $M_o \leq M_e$ if we have (6).

Theorem 3. We assume that the mode and median are within the same group, the m^{th} group. The necessary and sufficient condition for which $M_e \geq M_o$ (or $M_e \leq M_o$):

$$\sum_{i=1}^{m-a-1} f_i \geq (or \leq) \sum_{i=m+a+1}^{k} f_i$$

when

$$\sum_{i=m-a}^{m-1} f_i = \sum_{i=m+1}^{m+a} f_i,$$

and $\Delta_1 = \Delta_2$, where $a = 1, 2, \ldots, m-2$.

Proof. Assuming $M_e \leq M_o$, then we have

$$L + \frac{\Delta_1}{\Delta_1 + \Delta_2} d \leq L + \frac{\frac{n}{2} - F}{f_m} d.$$

But $L + \frac{\Delta_1}{\Delta_1+\Delta_2} d = L + \frac{1}{2} d$ since $\Delta_1 = \Delta_2$.

Additionally,

$$M_e = L + \frac{\frac{n}{2} - \sum_{i=1}^{m-1} f_i}{f_m} d$$

$$\frac{1}{2} \leq \frac{n - 2\sum_{i=1}^{m-1} f_i}{2 f_m}$$

$$f_m \leq n - 2\sum_{i=1}^{m-1} f_i$$

$$= \sum_{i=1}^{m-1} f_i + f_m + \sum_{i=m+1}^{k} f_i - 2\sum_{i=1}^{m-1} f_i$$

$$\text{since } n = \sum_{i=1}^{m-1} f_i + f_m + \sum_{i=m+1}^{k} f_i$$

$$\sum_{i=m+1}^{k} f_i - \sum_{i=1}^{m-1} f_i \geq 0$$

$$\left(\sum_{i=m+1}^{m+a} f_i + \sum_{i=m+a+1}^{k} f_i\right) - \left(\sum_{i=1}^{m-a-1} f_i + \sum_{i=m-a}^{m-1} f_i\right) \geq 0.$$

Hence

$$\sum_{i=m+a+1}^{k} f_i \geq \sum_{i=1}^{m-a-1} f_i$$

since

$$\sum_{i=m-a}^{m-1} f_i = \sum_{i=m+1}^{m+a} f_i.$$

On the other hand if we have

$$\sum_{i=m+a+1}^{k} f_i \geq \sum_{i=1}^{m-a-1} f_i$$

given the condition $\Delta_1 = \Delta_2$ and $\sum_{i=m-a}^{m-1} f_i = \sum_{i=m+1}^{m+a} f_i$, then we can prove that $M_e \leq M_o$. The rest of the theorem can be proved similarly.

Theorem 4. We assume that there are k groups and k is odd, the median is within the m^{th} group, $k = 2m - 1$. The necessary and sufficient condition for which $M_e \leq M$:

$$G - F \leq 2\, f_m\left(A - \frac{1}{2}(k+1)\right). \tag{7}$$

Proof. We claim that if $M_e \leq M$ then we have (7). First, notice that the median

$$\begin{aligned} M_e &= L + \frac{\frac{1}{2}n - F}{f_m} \times d \\ &= C + \left(\frac{k+1}{2} - 1\right)d - \frac{1}{2}d + \frac{\frac{1}{2}n - F}{f_m} \times d \\ &= C + \frac{k-2}{2}d + \frac{\frac{1}{2}n - F}{f_m}d \end{aligned}$$

and that the mean

$$M = C + \frac{\sum_{i=1}^{k}(i-1)\, f_i}{n}d.$$

Thus, if $M_e \leq M$, then we have

$$C + \frac{k-2}{2}d + \frac{\frac{1}{2}n - F}{f_m}d \leq C + \frac{\sum_{i=1}^{k}(i-1)\, f_i}{n}d$$

$$(k-2)\, f_m + F + f_m + G - 2F \leq 2\, f_m\frac{\sum_{i=1}^{k}(i-1)\, f_i}{n}$$

$$G - F \leq 2\, f_m\frac{\sum_{i=1}^{k}(i-1)\, f_i}{n} - (k-1)\, f_m$$

$$= \; 2 f_m (A - \frac{1}{2}(k+1)).$$

Conversely, if (7) holds, then we have $M_e \le M$.

Note that the median and the mean are almost always found in the same group. Nevertheless, it is not required to make this assumption in order to demonstrate Theorem 4.

Theorem 5. We assume that M_o and M_e are not within the same group, M_e is within the m^{th} group and M_o is within the $(m + 1)^{th}$ group. Then we have $M_e < M_o$. On the other hand, if we assume that M_o is within the m^{th} group and M_e is within the $(m + 1)^{th}$ group, then we have $M_e > M_o$.

Proof. Only the first half of the argument is shown by us since the second portion may be demonstrated in a similar manner.

Since

$$0 < \frac{\Delta_1}{\Delta_1 + \Delta_2} < 1,$$

we have

$$L < M_o = L + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times d < L + d = U,$$

where U is the upper boundary of the $(m + 1)^{th}$ group, or the modal group. On the other hand, since $F + f_m \ge \frac{1}{2}n$ and $f_m > 0$, we have $\frac{\frac{1}{2}n - F - f_m}{f_m} \le 0$.

Thus, $L + d(\frac{\frac{1}{2}n - F}{f_m} - 1) \le L.$ Therefore, $M_e \le L < M_o$.

Theorem 6. As in Theorem 5, we assume that M_o and M_e are not within the same group, M_e is within the m^{th} group and M_o is within the $(m + 1)^{th}$ group and we have k groups. Let

$$\alpha_1 = \frac{(2m-1) f_1 + (2m-3) f_2 + \cdots + 3 f_{m-1} + f_m}{\sum_{i=1}^{m} f_i}$$

$$\alpha_2 = \frac{f_{m+1} + 3 f_{m+2} + \cdots + [2(k-m) - 1] f_k}{\sum_{i=m+1}^{k} f_i}.$$

If $\alpha_1 \ge \alpha_2$ then we have $M < M_o$.

Proof. In order to demonstrate this, the formula for determining the mean is divided into two pieces.

$$M \; = \; \frac{1}{n}\Big\{ \sum_{i=1}^{m} f_i[L - \frac{1}{2}d - (m-i)d] + \sum_{i=m+1}^{k} f_i[L + \frac{1}{2}d + (i-(m+1))d] \Big\}$$

$$= \; L - \frac{d}{2n}\{(2m-1) f_1 + (2m-3) f_2 + \cdots + 3 f_{m-1} + f_m\} + \frac{d}{2n}\{ f_{m+1} + 3 f_{m+2} + \cdots + [2(k-m) - 1] f_k\}$$

$$= L - \frac{\alpha_1 d}{2n}\sum_{i=1}^{m} f_i + \frac{\alpha_2 d}{2n}\sum_{i=m+1}^{k} f_i$$

$$\leq L + \frac{\alpha_1 d}{2n}\left\{\sum_{i=m+1}^{k} f_i - \sum_{i=1}^{m} f_i\right\} \text{ since } \alpha_1 \geq \alpha_2$$

$$\leq L < M_o \text{ since } \left\{\sum_{i=m+1}^{k} f_i - \sum_{i=1}^{m} f_i\right\} \leq 0 \text{ and } M_o > L.$$

Note that, if $\alpha_1 \leq \alpha_2$ we cannot prove $M > M_o$ given other conditions.

In Theorem 6, it is presumed that the group containing the median comes immediately before the group containing the modal values. On the other hand, the median may be located anywhere, either before or after the modal group, particularly when there are more than two groups. Similar considerations may be used to the investigation of other situations. Comparisons of M_e and Mo were made in Theorem 5, while comparisons of M and Mo were made in Theorem 6. Utilising Theorem 4 allows one to ascertain the nature of the connection that exists between M and M_e.

In Table 3, we have compiled a summary of our findings pertaining to theorems 1 through 4.

Table 3. Conditions for mean-median-mode inequalities

Condition	Conclusion
(1) $f_m\Delta_r \leq G - F \leq 2 f_m(A - \frac{1}{2}(k+1))$	$M_o \leq M_e \leq M$
(2) $G - F \geq 2 f_m(A - \frac{1}{2}(k+1))$ or $G - F \leq f_m\Delta_r$	$M \leq M_e \leq M_o$
(3) $G - F \geq f_m\Delta_r, A \leq \Delta_r^* + \frac{k}{2}$	$M \leq M_o \leq M_e$
(4) $G - F \leq f_m\Delta_r, A \geq \Delta_r^* + \frac{k}{2}$	$M_e \leq M_o \leq M$
(5) $G - F \geq 2 f_m(A - \frac{1}{2}(k+1)), A \geq \Delta_r^* + \frac{k}{2}$	$M_o \leq M \leq M_e$
(6) $G - F \leq 2 f_m(A - \frac{1}{2}(k+1)), A \leq \Delta_r^* + \frac{k}{2}$	$M_e \leq M \leq M_o$

The distinction that exists between F, the cumulative frequency of the groups that come before the median or modal group, and G, the cumulative frequency of the groups that come after the median or modal group, plays a significant part in the way that the mean M, M_e, and the mode M_o are related to one another.

Let

$$\delta_1 = n - 2F - f_m - A + \Delta_r^* + \frac{k}{2}, \delta_2 = f_m\Delta_r, \delta_3 = 2 f_m(A - \frac{1}{2}(k+1)), \Delta = G - F.$$

If we suppose that $\delta_1 \leq \delta_2 \leq \delta_3$, then Figure 1 illustrates the connection between the mean value M, the median value M_e, and the mode value M_o. One can observe from Figure 1 that the mode is the largest and that the mean is greater than the median if $\Delta \leq \delta_1$, that the mean is larger than the mode, which is larger than the median if $\delta_1 \leq \Delta \leq \delta_2$, that the mean is larger than the median, which is larger than the mode if $\delta_2 \leq \Delta \leq \delta_3$, and that the median is larger than the mean, which is larger than the mode if $\Delta \geq \delta_3$, respectively. Take note that the connection $\delta_1 \leq \delta_2 \leq \delta_3$ is only one of many potential outcomes that might occur. Given a different sample of grouped data, it is possible for it to be in an entirely different order.

5. Concluding Remarks

Within the context of this research article, we investigated the degree to which the mean, median, and mode of two asymmetric unimodal continuous distributions are correlated with one another. When working with grouped data, our objective was to investigate the nature of the connection between these measures and determine the conditions—necessary and/or sufficient—that must be met for any hypothetical mean-median-mode inequalities to be considered genuine. Throughout the course of our investigation, we have categorised the data and made a number of assumptions pertaining to the inequalities. These assumptions include the presence of a common width for each group (which is applied to all theorems), the mode and median being located within the same group (Theorems 2 and 3), the mode and median not being located within the same group (Theorems 5 and 6), and the assumption of k groups with an odd value of k, where the median and mode are in the m^{th} group, with $k = 2m 1$ (Theorems 1 and 4). When it comes to dealing with facts from the actual world, however, we understand that these presumptions may not always be accurate. Data that has been gathered together might have a very different presentation based on a number of aspects, including the design of the survey, the data collecting, the data input, the cleaning, the coding, and the analysis. As a consequence of this, it would be required to amend the theorems stated in this work correspondingly in the event that the data did not conform to one or more of the aforementioned assumptions.

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