2-FIBONACCI SEQUENCES APPLYING THE CONCEPTS OF CONGRUENCE AND MATRIX METHODS

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Accepted: 13.03.2022 **Published**: 01.04.2022

Keywords: 2-Fibonacci sequences, Binet's formula, congruence, Pentanacci Sequences, Generating Function.

Abstract

In this paper, we have obtained the generating function and a general formula to find out nth term of 2-Fibonacci sequences. In the later section, some generalized properties of 2-Fibonacci sequences are finding out by applying the concept of congruence and matrix methods.

Paper Identification

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1. Introduction

In [1], four different ways of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are described and are called 2-Fibonacci sequences. Many properties were investigated by many authors like [1,2,3,4,5,6]. In this paper, we consider the following 2-Fibonacci sequences defined as:

$$\alpha_{n+2} = \beta_{n+1} + \beta_n, \quad \beta_{n+2} = \alpha_{n+1} + \alpha_n; \; n = 0,1,2,\dots \quad (1)$$

with initial conditions $\alpha_0 = 0$, $\alpha_1 = 1$, $\beta_0 = 2$ and $\beta_1 = 1$. First few terms of 2-Fibonacci sequences are given by

$\alpha_2 = 3$, $\beta_2 = 1$

$\alpha_3 = 2$, $\beta_3 = 4$

$\alpha_4 = 5$, $\beta_4 = 5$

$\alpha_5 = 9$, $\beta_5 = 7$

and so on.

Now, the following equation

$$r^4 - r^2 - 2r - 1 = 0 \quad (2)$$

is satisfied by the above 2-Fibonacci sequences.

On solving equation (2), the four roots are given by

$$\gamma = \frac{1+\sqrt{5}}{2}, \delta = \frac{1-\sqrt{5}}{2}, \omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

From the recurrence relation we observed that 2-Fibonacci numbers are also forms a complete sequence. This means that every positive integer can be written as a sum of 2-Fibonacci numbers, where any one number is used once at most. Also, *by taking the ratio of a two consecutive 2-Fibonacci numbers larger divided by smaller, the sequence obtained approaches to the golden ratio.*

2. *Generating Functions*

Let $g_{F,L}(t)$ *be a polynomial of infinite degree with coefficients as 2-Fibonacci sequence* $\{\alpha_n\}$ *(defined by (1)) i.e., defined by*

$$g_{F,L}(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

Theorem 2.1 Then the generating functions for 2-Fibonacci sequences $\{\boldsymbol{\alpha_n}\}$ ***and*** $\{\beta_n\}$ ***are given by***

$$\sum_{n=0}^{\infty} \alpha_n t^n = \frac{t + 3t^2 + t^3}{1 - t^2 - 2t^3 - t^4}$$

and

$$\sum_{n=0}^{\infty} \beta_n t^n = \frac{2 + t - t^2 - t^3}{1 - t^2 - 2t^3 - t^4}$$

Proof *Consider*

$$g_{F,L}(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

$$= t + 3t^2 + 2t^3 + \sum_{n=4}^{\infty} (\alpha_{n-2} + 2\alpha_{n-3} + \alpha_{n-4}) t^n$$

$$= t + 3t^2 + t^3 + g_{F,L}(t)(t^2 + 2t^3 + t^4)$$

i.e

$$g_{F,L}(t) = \frac{t + 3t^2 + t^3}{1 - t^2 - 2t^3 - t^4}$$

Similarly, generating function $g_{F,L}(t)$ *for 2-Fibonacci sequence* $\{\beta_n\}$

$$\sum_{n=0}^{\infty} \beta_n t^n = \frac{2 + t - t^2 - t^3}{1 - t^2 - 2t^3 - t^4}$$

Theorem 2.1 For all non-negative integer n, the nth term of 2-Fibonacci sequences can be expressed by following formulas

$$\boldsymbol{\alpha_n} = \frac{(\gamma^{n+4} + \gamma^{n+1})}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{(\delta^{n+4} + \delta^{n+1})}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)} + \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} + \frac{2\omega^{2n+2}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)}$$

and

$$\boldsymbol{\beta_n} = \frac{2\gamma^{n+3}}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{2\delta^{n+3}}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)} - \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} - \frac{2(\omega^2)^{n+1}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)}$$

Proof

We can easily prove this theorem by using the generating function for the 2-Fibonacci sequences α_n *and* β_n *respectively.*

3. *Some Results of 2-Fibonacci Sequences based on Congruence*

3.1 $\boldsymbol{\alpha_n \equiv (\beta_n + (1 - r))(mod\ 3)}$, ***where*** $\boldsymbol{n \equiv r(mod\ 3)}$

Proof *The result being certainly true for* $n = 1$*.Let us assume that results holds for* $n = 2,3, \ldots k$*. Now, by using induction and* (1)*, we have*

$$\beta_{k+1} + (1 - r) \equiv \beta_k + \beta_{k-1} - 3r + 6(mod\ 3) \equiv \beta_k + \beta_{k-1}(mod\ 3) \equiv \alpha_{k+1}(mod\ 3).,$$

where $k \equiv r - 1(mod\ 3)$*. Thus, by induction it holds for all natural number*

Corollary 3.1.1 $\boldsymbol{\alpha_n > \beta_n}$ ***, if*** $\boldsymbol{n \equiv 2(mod\ 3)}$ ***and*** $\boldsymbol{\beta_n > \alpha_n}$ ***, if*** $\boldsymbol{n \equiv 0(mod\ 3)}$

Proof *It follows directly from (3.1)*

3.2 Relation of 2-Fibonacci sequences with Fibonacci sequence

$\boldsymbol{F_{n+1} \equiv \alpha_n + (1-r) \equiv \beta_n + (r-1)(mod\ 3)}$.*where* $\boldsymbol{n \equiv r(mod\ 3)}$ $\quad (3)$

Proof *Clearly, result holds for* $n = 1$.

Now, by using induction and result (3.1), we have

$$F_{n+2} = F_{n+1} + F_n \equiv \beta_{n+1} + (r-1)(mod\ 3)$$
$$\equiv \alpha_{n+1} + (1-r)(mod\ 3),$$

where $n + 1 \equiv r(mod\ 3)$.

Thus, by induction, (3) is true for all natural number

3.3 $\boldsymbol{\alpha_n}$ ***and*** $\boldsymbol{\beta_n}$ ***is even if*** $\boldsymbol{n \equiv 0(mod\ 3)}$ ***and it is odd if*** $\boldsymbol{n \equiv 1, 2(mod\ 3)}$

Proof *From [5],* F_n *is even if* $n \equiv 0(mod\ 3)$, *therefore from relation (3), the above result holds*

3.4 Binet Formula for 2-Fibonacci sequences can also be expressed as

$$\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} = \alpha_n + (1-r) = \beta_n + (r-1),$$

where $n \equiv r(mod\ 3)$ $\quad (4)$

Proof *The Binet formula for Fibonacci sequence [3] is given by*

$$\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} = F_{n+1}$$

where $\gamma = \frac{1+\sqrt{5}}{2}$, $\delta = \frac{1-\sqrt{5}}{2}$,*therefore (3.2) follows (3.4)*

3.5 If $\boldsymbol{n/3}$, ***then*** $\boldsymbol{{\beta_n}^2 \equiv 0(mod\ 4)}$ ***,otherwise*** $\boldsymbol{{\beta_n}^2 \equiv 1(mod\ 4)}$

3.6 If $\boldsymbol{n/3}$, ***then*** $\boldsymbol{{\alpha_n}^2 \equiv 0(mod\ 4)}$ ***,otherwise*** $\boldsymbol{{\alpha_n}^2 \equiv 1(mod\ 4)}$

3.7 $\boldsymbol{{\beta_n}^2 - {\beta_{n-3}}^2 \equiv {\alpha_n}^2 - {\alpha_{n-3}}^2(mod\ 4)}$

Proof *3.5 and 3.6 directly follows from result 3.3.Also,the result 3.7 is particular case of 3.5 and 3.6*

3.8 (Cassini identity) $\boldsymbol{\beta_{n-1}\beta_{n+1} - {\beta_n}^2 \equiv 3(-1)^n(mod\ 4)}$

3.9 $\boldsymbol{\alpha_{n-1}\alpha_{n+1} - {\alpha_n}^2 \equiv 3(-1)^{n-1}(mod\ 4)}$

Proof

Clearly, result holds for $n = 1$.*Let result hold for n.*

By using 3.7 and (1), *consider*

$$\beta_n\beta_{n+2} - {\beta_{n+1}}^2 = \beta_n(\beta_n + 2\beta_{n-1} + \beta_{n-2}) - (\beta_{n-1} + 2\beta_{n-2} + \beta_{n-3})^2$$
$$\equiv {\beta_n}^2 - {\beta_{n-3}}^2 + 2\beta_n\beta_{n-1} + \beta_n\beta_{n-2} - {\beta_{n-1}}^2 - 2\beta_{n-1}\beta_{n-3}(mod\ 4)$$
$$\equiv 2\beta_{n-1}(\beta_n - \beta_{n-3)} + 3(-1)^{n-1}(mod\ 4)$$
$$\equiv 3(-1)^{n+1}(mod\ 4)$$

Hence, the result, in a similar way we can prove 3.9 identity

4. *Results using Matrix Methods*

Introduce a matrix $E = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ *such that*

$detE = -1$.*Here, matrix E is called generating matrix for 2-Fibonacci sequences defined by equation (1).*

Theorem 4.1 For all positive integers $\boldsymbol{n}$ ***following results hold:***

(a) $$\begin{pmatrix} \alpha_{n+3} \\ \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \\ \alpha_{n-1} \end{pmatrix}$$

(b) $$\begin{pmatrix} \beta_{n+3} \\ \beta_{n+2} \\ \beta_{n+1} \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_{n+2} \\ \beta_{n+1} \\ \beta_n \\ \beta_{n-1} \end{pmatrix}$$

(c) $$\begin{pmatrix} \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \\ \alpha_{n-1} \end{pmatrix} = E^{n-1} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

(d) $$\begin{pmatrix} \beta_{n+2} \\ \beta_{n+1} \\ \beta_n \\ \beta_{n-1} \end{pmatrix} = E^{n-1} \begin{pmatrix} 4 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

Proof *By using induction, these results have a simple proof.*

Now, we will establish a formula named as Binet's formula for finding out any nth term of 2-Fibonacci sequences using the matrix E. [8, 9] established the Binet's formula for Tribonacci and Pentanacci sequence. Now, the characteristic polynomial of matrix E is given by

$$|E-\lambda I_{4\times4}| = \begin{vmatrix} -\lambda & 1 & 2 & 1 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = \lambda^4-\lambda^2-2\lambda-1$$

Where λ is the eigenvalue of the matrix E. On solving the characteristic equation, the four eigen values of matrix E are given by

$$\gamma = \frac{1+\sqrt{5}}{2}, \delta = \frac{1-\sqrt{5}}{2}, \omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}.$$

Next, we will find out the eigenvectors corresponding to these eigenvalues. The eigenvectors are the non-zero solutions of the following homogenous system of equation

$$(E-\lambda I_{4\times4})y = 0$$

where y is a column vector of order 4×1.By solving this homogenous system of equations, eigen vectors corresponding to eigenvalues γ, δ, ω and ω^2 are $\begin{pmatrix} \gamma^3 \\ \gamma^2 \\ \gamma \\ 1 \end{pmatrix}, \begin{pmatrix} \delta^3 \\ \delta^2 \\ \delta \\ 1 \end{pmatrix}, \begin{pmatrix} \omega^3 \\ \omega^2 \\ \omega \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} (\omega^2)^3 \\ (\omega^2)^2 \\ \omega^2 \\ 1 \end{pmatrix}$ *respectively.*

Now, consider a matrix M containing eigenvectors of matrix E

$$M = \begin{pmatrix} \gamma^3 & \delta^3 & \omega^3 & \omega^6 \\ \gamma^2 & \delta^2 & \omega^2 & \omega^4 \\ \gamma & \delta & \omega & \omega^2 \\ 1 & 1 & 1 & 1 \end{pmatrix}_{4\times4}$$

Then its inverse is given by

$$M^{-1} = \frac{1}{det\, M} \begin{pmatrix} A & A(\omega^2+\omega+\delta) & -\omega A(\omega^2+\omega\delta+\delta) & \delta\omega\omega^2 A \\ -B & -B(\omega^2+\omega+\gamma) & \omega B(\omega^2+\omega\gamma+\gamma) & -\gamma\omega\omega^2 B \\ C & C(\omega^2+\gamma+\delta) & -C(\omega^2\gamma+\omega^2\gamma+\gamma\delta) & \gamma\delta\omega^2 C \\ -D & -D(\omega+\gamma+\delta) & D(\omega\gamma+\gamma\delta+\omega\delta) & -\gamma\delta\omega D \end{pmatrix}_{4\times4}$$

where $A = (\omega-\delta)(\omega^2-\delta)(\omega^2-\omega)$, $B = (\omega-\gamma)(\omega^2-\gamma)(\omega^2-\omega)$,
$C = (\delta-\gamma)(\omega^2-\gamma)(\omega^2-\delta)$ *and* $det\, M = (\omega-\delta)(\omega^2-\delta)(\omega^2-\omega)(\omega-\gamma)(\omega^2-\gamma)(\gamma-\delta)$.

Let's consider a diagonal matrix D whose diagonal entries are the eigenvalues of matrix E. Then using the diagonalizablity of matrix E, we have $E = MDM^{-1}$ or $E^n = MD^nM^{-1}$.Then, by using theorem 3.1 and the recurrence relation of 2-Fibonacci numbers, we have

$$\alpha_n = \frac{(\gamma^{n+4}+\gamma^{n+1})}{(\gamma-\delta)(\gamma-\omega)(\gamma-\omega^2)} + \frac{(\delta^{n+4}+\delta^{n+1})}{(\delta-\gamma)(\delta-\omega)(\delta-\omega^2)} + \frac{2\omega^{n+1}}{(\omega-\gamma)(\omega-\delta)(\omega-\omega^2)} + \frac{2\omega^{2n+2}}{(\omega^2-\gamma)(\omega^2-\delta)(\omega^2-\omega)}$$

And

$$\beta_n = \frac{2\gamma^{n+3}}{(\gamma-\delta)(\gamma-\omega)(\gamma-\omega^2)} + \frac{2\delta^{n+3}}{(\delta-\gamma)(\delta-\omega)(\delta-\omega^2)} - \frac{2\omega^{n+1}}{(\omega-\gamma)(\omega-\delta)(\omega-\omega^2)} - \frac{2(\omega^2)^{n+1}}{(\omega^2-\gamma)(\omega^2-\delta)(\omega^2-\omega)}$$

Hence proved.

RÉFÉRENCIAS

1. Atanassov K.T., Atanassova L. & Sasselov D.D. A new perspective to the generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 23(1)(1985), pp.21-28.
2. Atanassov, K.T. On a second new generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 24(4)(1986), pp.362-365.
3. Atanassov, K.T. Combined 2-Fibonacci sequences. *Notes on Number Theory and Discrete Mathematics*, 16(1)(2010), pp.24-28.
4. Atanassov, K. T. Combined 2-Fibonacci sequences. Part 2. *Notes on Number Theory and Discrete Mathematics*, 16(4)(2010), pp.18-24.
5. Atanassov, K. T. *New visual perspectives on Fibonacci numbers*. World Scientific, 2002.
6. Lee, J.Z & Lee, J.S. Some properties of the generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 25(2)(1987), pp.111-117.
7. Soykan, Y& Okumus, I. On a generalized tribonacci sequence. *Journal of Progressive Research in Mathematics*, 14(3)(2019), pp.2413-18.
8. Soykan, Y. On a generalized pentanacci sequence. *Asian Research Journal of Mathematics*, 14(3)(2019), pp. 1-9.