APPROXIMATION OF POLYNOMIAL FUNCTIONS USING WAVELET BASES

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Abstract

There are numerous approaches in numerical mathematics for approximating a given function by a class of simpler functions. Recently, wavelet functions have been demonstrated to be an efficient approximation tool. The writers of this article have demonstrated how to approximate a polynomial function using wavelet bases. Thus, this work is explanatory in nature and includes a full description of the procedure.

Paper Identification

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1. Introduction

In this Paper,we consider the problem of approximating a given function with the help of a wavelet function. Since the polynomial functions are commonly used in numerical mathematics, let us investigate the procedure for their approximation using wavelet techniques. In chapter-7 of [3] Rafael C.Gonzalez and Richard E. Woods have shown that a signal or function $f(x)$ can be better analyzed if we express it as a linear combination of expansion functions

$$f(x) = \sum_k a_k \phi_k(x)$$

where k is an integer, a_k are real-valued expansion cofficients and the $\phi_k(x)$ are expansion functions. If the expansion set$\{\phi_k(x)\}$ is a basis, for the class of functions that can be so expressed, then the expansion is unique.The expressible functions form a function space.This can be denoted as

$$V = \overline{span_k \phi_k(x)}$$

$f(x) \in V$ means that $f(x)$ is in the closed span of $\{\phi_k(x)\}$. The expansion functions may form an orthonormal basis for V or it may be a frame.

1.1 The wavelet series expansions

In this work, wavelet functions have been used as expansion functions. Consider integer translations and binary dilation of a scaling function $\phi(x)$, that is,

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$$

We can select $\phi(x)$ wisely so that $\{\phi_{j,k}(x)\}$ can be made to span the set of all measurable, square-integrable functions $L^2(R)$.

If we specify the value $j = j'$, the resulting expansion set,$\{\phi_{j',k}(x)\}$ is a subset of $\{\phi_{j,k}(x)\}$.

We can define that subspace as

$$V_{j'} = \overline{span_k \phi_{j',k(x)}}$$

and $f(x) \in V_{j'}$ can be written as

$$f(x) = \sum_k a_k \phi_{j',k}(x)$$

In general, we will denote the subspace spanned over k for any j as

$$V_j = \overline{span_k \phi_{j,k}(x)} \qquad (1)$$

For a scaling function that satisfies Multiresolution Analysis (MRA) conditions, we can define a wavelet function $\psi(x)$ that, together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces , V_j and V_{j+1}.

We denote the set of wavelets $\{\psi_{j,k}(x)\}$ as

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$$

Thus

$$V_1 = V_0 \oplus W_0$$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1 \qquad (2)$$

We write

$$W_j = \overline{span\psi_{j,k}(x)}$$

For $f(x) \in W_j$,

$$f(x) = \sum_k a_k \psi_{j,k}(x)$$

The Scaling and wavelet function spaces are related by

$$V_{j+1} = V_j \oplus W_j$$

where $\oplus$ denote the union of spaces. The orthogonal compliment of V_j in V_{j+1} is W_j and all members of V_j are orthogonal to the members of W_j. Therefor,

$$< \phi_{j,k}, \psi_{j,l} >= 0$$

for $j, k, l \in Z$

Thus, we can express the space of all measurable square-integrable functions as

$$L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots. \qquad (3)$$

or

$$L^2(R) = V_1 \oplus W_1 \oplus W_2 \oplus \ldots.$$

or

$$L^2(R) = \ldots\ldots\oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots..$$

In general ,for an arbitary starting scale j' , we can write

$$L^2(R) = V_{j'} \oplus W_{j'} \oplus W_{j'+1} \oplus \ldots..$$

From (1),(2) and (3), it follows that

$$\{\psi_{j,k}: j,k \in Z, j \geq 0\} \cup \{\phi_{0,k}: k \in Z\} \qquad (4)$$

is a basis for $L^2(R)$. Hence, we define the wavelet series expansion of function $f(x) \in L^2(R)$ relative to wavelet $\psi(x)$ and scaling function $\phi(x)$. Using (4) we can write wavelet series expansion of function $f(x) \in L^2(R)$

$$f(x) = \sum_k c_0(k)\phi_{0,k}(x) + \sum_{j=0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x) \qquad (5)$$

where $c_o(k)$ are called approximation or scaling cofficients, and $d_j(k)$ are termed as wavelet cofficients. These expansion coefficients are calculated as

$$c_0(k) =< f(x), \phi_{0,k}(x) >= \int f(x)\phi_{0,k}(x)dx \qquad (6)$$

and

$$d_j(k) =< f(x), \psi_{j,k}(x) >= \int f(x)\psi_{j,k}dx \qquad (7)$$

One can go through the texts [1] ,[2] and [4] for detailed study.

2. Procedure of Approximation

In order to illustrate the procedure, let us consider the quadratic function

$$f(x) = \begin{cases} x^2 - 5x + 6, & 0 \le x < 4 \\ 0, & elsewhere. \end{cases}$$

shown in Fig.(1). For the sake of simplicity, we are using Haar wavelets.

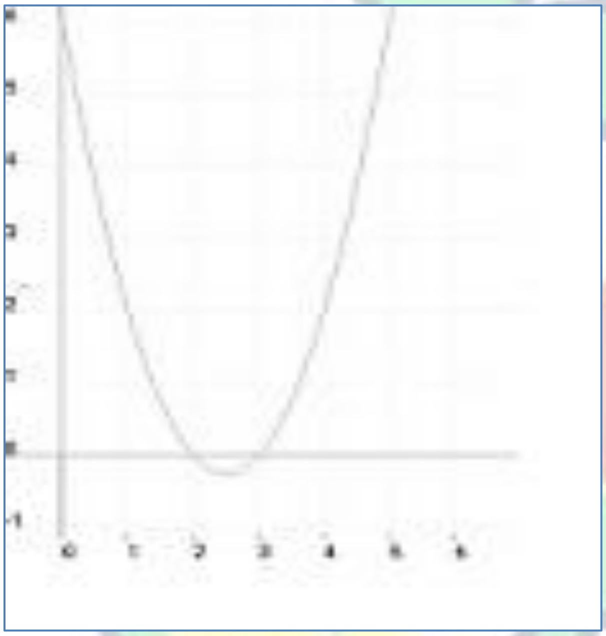

Figure 1: Graph of f(x)

Haar Scaling function is defined as

$$\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0 & elsewhere. \end{cases}$$

Haar wavelet function is

$$\psi(x) = \begin{cases} 1, & 0 \le x < 0.5 \\ -1, & o.5 \le x < 1) \\ 0, & elsewhere. \end{cases}$$

From equation (5)

.

$$y = f(x) = \sum_k c_0(k)\phi_{0,k}(x) + \sum_{j=0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x)$$

$$y = f(x) = c_0(0)\phi_{0,0}(x) + c_0(1)\phi_{0,1}(x) + c_0(2)\phi_{0,2}(x) + c_0(3)\phi_{0,3}(x) + \sum_{j=0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x)$$

Equation (6) can be used to compute expansion coefficients $c_0(k)$:

$$c_0(0) = \int_0^4 (x^2 - 5x + 6)\phi_{0,0}(x)dx = \int_0^1 (x^2 - 5x + 6)dx = \frac{23}{6} = 3.833$$

$$c_0(1) = \int_0^4 (x^2 - 5x + 6)\phi_{0,1}(x)dx = \int_0^4 (x^2 - 5x + 6)\phi(x-1)dx = \int_1^2 (x^2 - 5x + 6)dx = \frac{5}{6} = 0.833$$

$$c_0(2) = \int_0^4 (x^2 - 5x + 6)\phi_{0,2}(x)dx = \int_2^3 (x^2 - 5x + 6)dx = \frac{-1}{6} = -0.166$$

$$c_0(3) = \int_0^4 (x^2 - 5x + 6)\phi_{0,3}(x)dx = \int_3^4 (x^2 - 5x + 6)dx = \frac{5}{6} = 0.833$$

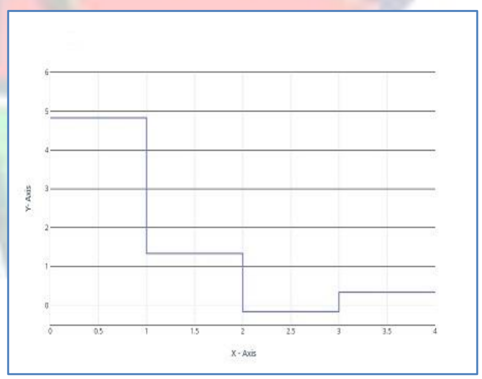

Figure 2: Graph of V_0

Using the above expansion coefficients Fig (2) shows the approximation of the function in the subspace V_0.

The wavelet coefficients $d_j(k)$ are computed using equation (7). At level j= 0,

$$d_0(0) = \int_0^4 (x^2 - 5x + 6)\psi_{0,0}(x)dx$$

$$= \int_0^1 (x^2 - 5x + 6)\psi(x)dx$$

$$= \int_0^{\frac{1}{2}} (x^2 - 5x + 6)\psi(x)dx + \int_{\frac{1}{2}}^1 (x^2 - 5x + 6)\psi(x)dx$$

$$= \int_0^{\frac{1}{2}} (x^2 - 5x + 6)dx - \int_{\frac{1}{2}}^1 (x^2 - 5x + 6)dx$$

$$= 1$$

$$d_0(1) = \int_0^4 (x^2 - 5x + 6)\psi_{0,1}(x)dx$$

$$= \int_0^4 (x^2 - 5x + 6)\psi(x-1)dx$$

$$= \int_1^{\frac{3}{2}} (x^2 - 5x + 6)\psi(x)dx + \int_{\frac{3}{2}}^2 (x^2 - 5x + 6)\psi(x)dx$$

$$= \int_1^{\frac{3}{2}} (x^2 - 5x + 6)dx - \int_{\frac{3}{2}}^2 (x^2 - 5x + 6)dx$$

$$= \frac{1}{2} = 0.5$$

$$d_0(2) = \int_0^4 (x^2 - 5x + 6)\psi_{0,2}(x)dx$$

$$= \int_0^4 (x^2 - 5x + 6)\psi(x-2)dx$$

$$= \int_2^{\frac{5}{2}} (x^2 - 5x + 6)\psi(x)dx + \int_{\frac{5}{2}}^3 (x^2 - 5x + 6)\psi(x)dx$$

$$= \int_2^{\frac{5}{2}} (x^2 - 5x + 6)dx - \int_{\frac{5}{2}}^3 (x^2 - 5x + 6)dx$$

$$= 0$$

$$d_0(3) = \int_0^4 (x^2 - 5x + 6)\psi_{0,3}(x)dx$$

$$= \int_0^4 (x^2 - 5x + 6)\psi(x-3)dx$$

$$= \int_3^{\frac{7}{2}} (x^2 - 5x + 6)\psi(x)dx + \int_{\frac{7}{2}}^4 (x^2 - 5x + 6)\psi(x)dx$$

$$= \int_3^{\frac{7}{2}} (x^2 - 5x + 6)dx - \int_{\frac{7}{2}}^4 (x^2 - 5x + 6)dx$$

$$= \frac{-1}{2} = -0.5$$

using the above wavelet coefficients Fig. (3) shows the approximation of the function in the subspace W_0.

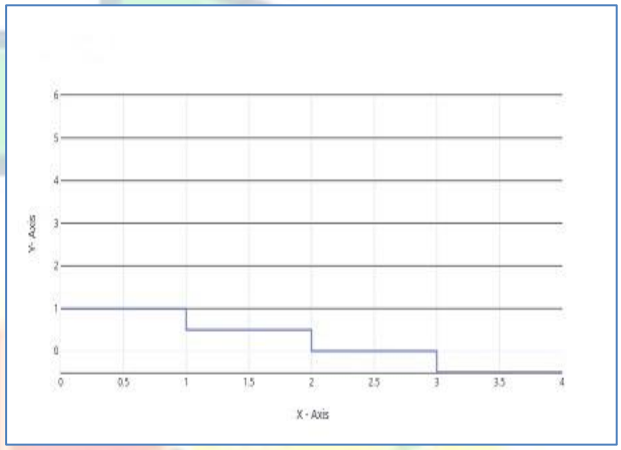

Figure 3: Graph of W_0

Since

$$V_1 = V_0 \oplus W_0$$

Fig.(4) shows the approximation of the function in the subspace V_1.

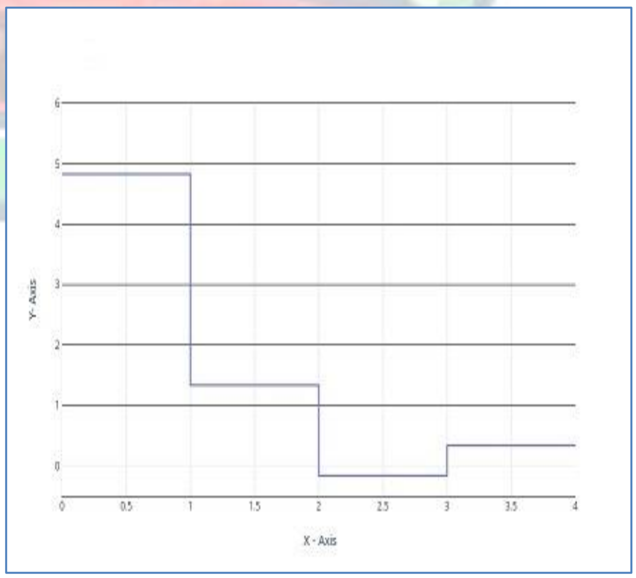

Figure 4: Graph of V_1

At level j=1 , let us compute wavelet coefficients.

$d_1(0) = \int_0^4 (x^2 - 5x + 6)\psi_{1,0}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x)dx$

$= \int_0^{\frac{1}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{1}{4}}^{\frac{1}{2}} \sqrt{2}(x^2 - 5x + 6)dx$

$= \frac{54\sqrt{2}}{192} = 0.3977$

$d_1(1) = \int_0^4 (x^2 - 5x + 6)\psi_{1,1}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x - 1)dx$

$= \int_{\frac{1}{2}}^{\frac{3}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{3}{4}}^{1} \sqrt{2}(x^2 - 5x + 6)dx$

$= \frac{42\sqrt{2}}{192} = 0.3093$

$d_1(2) = \int_0^4 (x^2 - 5x + 6)\psi_{1,2}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x - 2)dx$

$= \int_1^{\frac{5}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{5}{4}}^{\frac{3}{2}} \sqrt{2}(x^2 - 5x + 6)dx$

$= \frac{30\sqrt{2}}{192} = 0.2209$

$d_1(3) = \int_0^4 (x^2 - 5x + 6)\psi_{1,3}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x - 3)dx$

$= \int_{\frac{3}{2}}^{\frac{7}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{7}{4}}^{2} \sqrt{2}(x^2 - 5x + 6)dx$

$= 0.1325$

$d_1(4) = \int_0^4 (x^2 - 5x + 6)\psi_{1,4}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x - 4)dx$

$= \int_2^{\frac{9}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{9}{4}}^{\frac{5}{2}} \sqrt{2}(x^2 - 5x + 6)dx$

$= \frac{6\sqrt{2}}{192} = 0.0441$

$d_1(5) = \int_0^4 (x^2 - 5x + 6)\psi_{1,5}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x - 5)dx$

$= \int_{\frac{5}{2}}^{\frac{11}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{11}{4}}^{3} \sqrt{2}(x^2 - 5x + 6)dx$

$= \frac{-6\sqrt{2}}{192} = -0.0441$

$d_1(6) = \int_0^4 (x^2 - 5x + 6)\psi_{1,6}(x)dx$

$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x - 6)dx$

$= \int_3^{\frac{13}{4}} (x^2 - 5x + 6)\sqrt{2}dx - \int_{\frac{13}{4}}^{\frac{7}{2}} \sqrt{2}(x^2 - 5x + 6)dx$

$= \frac{-18\sqrt{2}}{192} = -0.1325$

$d_1(7) = \int_0^4 (x^2 - 5x + 6)\psi_{1,7}(x)dx$

$$= \int_0^4 (x^2 - 5x + 6)\sqrt{2}\psi(2x-7)dx$$

$$= \int_{\frac{7}{2}}^{\frac{15}{4}} (x^2 - 5x + 6)\sqrt{2}dx -$$

$$\int_{\frac{15}{4}}^{4} \sqrt{2}(x^2 - 5x + 6)dx$$

$$= \frac{-30\sqrt{2}}{192} = -0.2209$$

The above wavelet coefficients have been used to approximate the function in the subspace of next scale. Fig.(5) shows the approximation of the function in the subspace W_1.

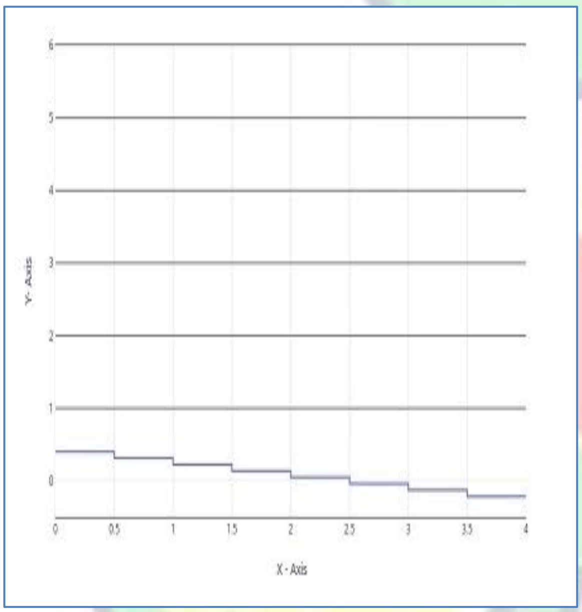

Figure 5: Graph of W_1

$$V_2 = V_1 \oplus W_1$$

$$c_0(0) + d_0(0) + d_1(0) = 5.2307$$

$$c_0(0) + d_o(0) + d_1(1) = 5.1423$$

$$c_0(1) + d_0(1) + d_1(2) = 1.5539$$

$$c_0(1) + c_0(1) + d_1(3) = 1.4655$$

$$c_0(2) + c_0(2) + d_1(4) = -0.1219$$

$$c_0(2) + c_0(2) + d_1(5) = -0.2101$$

$$c_0(3) + c_0(3) + d_1(6) = 0.2005$$

$$c_0(3) + c_0(3) + d_1(7) = 0.1121$$

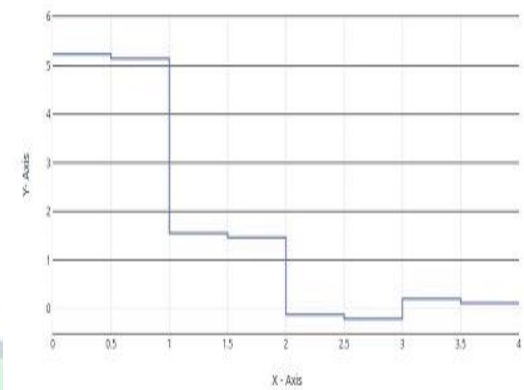

Figure 6: Graph of V_2

These values have been used for next scale approximation. Fig.(6) shows the approximation of the function in the subspace V_2.

3. Conclusion

The terms $c_0(k)$ are being used to generate a subspace V_0 approximation of the function. This approximation is shown in Fig(2) and is the average value of the original function. The terms $d_o(k)$ are used to refine the approximation by adding a level of detail from subspace W_0. The added detail and resulting V_1 approximation are shown in Fig.(3) and Fig.(4), respectively. Another level of detail is added by the subspace W_1 coefficients $d_1(k)$. This additional detail is shown in Fig.(5) and the resulting V_2 approximation is defined in Fig.(6).Note that the expansion is now beginning to resemble the original function. As higher scales are added, the approximation becomes a more precise representation of the function,realizeing it in the limit as $j \to \infty$. It is also important to note that we are using Haar wavelet bases which are itself discontinuous and approximating a continuous smooth function.Therefore, the resulting approximation is not well effective. Instead, if we use spline wavelets, then the resulting approximation is expected to be more effective.

RÉFÉRENCIAS

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