SIGNIFICANCE AND PRACTICAL APPLICATIONS OF FUZZY SET EXTENSIONS

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Abstract

Fuzzy sets are a powerful tool for coping with uncertainty and imprecision in a variety of disciplines, but their basic implementations are limited in their ability to represent complex and ambiguous data. This paper investigates the significance and practical applications of fuzzy set extensions, such as Intuitionistic Fuzzy Sets, Pythagorean Fuzzy Sets, and Fermatean Fuzzy Sets, which surmount these limitations and permit more complex analysis. In addition, we discuss operators on Intuitionistic Fuzzy Sets, establish theorems regarding their relationships, and introduce a new distance measure that takes into account both membership and non-membership functions, demonstrating its significance through a pattern recognition problem. The results demonstrate the potential of fuzzy set extensions, operators, and distance measures for obtaining a deeper understanding of complex real-world systems and making informed decisions in a variety of fields.

Paper Identification

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1. Introduction

L.A. Zadeh created fuzzy set (FS) theory in 1965{15} to resolve ambiguous and inaccurate information. Each entry in a FS has a membership value, which indicates the degree of an event and has a value between [0,1]. Numerous decision-making issues can be solved with fuzzy sets, including medical diagnosis, pattern identification, cluster analysis and many others. Atanassov {1} thought up the I-FS (intuitionistic fuzzy set). Each I-FS element has a M-D (membership degree) and a N-MD (non-membership degree) in the range [0,1] having sum less than or equal to 1. This limit on the total of M-D limits the application of I-FSs. Yeger {13} proposed the concept of P-FS (Pythagorean fuzzy set) as an extension of I-FSs. Every element in a P-FS has a M-G of $h_A(\mathscr{s})$ and a N-MG of $g_A(\mathscr{s})$ with the square sum of these two grades being no more than one, $\left(h_A(\mathscr{s})\right)^2 + \left(g_A(\mathscr{s})\right)^2 \geq 1$. For instance, if $h_A(\mathscr{s}) = 0.8$ and $g_A(\mathscr{s}) = 0.7$ then $\left(h_A(\mathscr{s})\right)^2 + \left(g_A(\mathscr{s})\right)^2 = 1.13 > 1$. Senapati and Yager {11} then put out the idea of F-FSs (fermatean fuzzy sets). A F-FS has the following properties: $\left(r_f(\mathscr{s})\right)^3 + \left(s_f(\mathscr{s})\right)^3 \leq 1$. This suggests that F-FSs are more powerful than FSs, I-FSs, and P-FSs since they are all confined within the space of F-FSs. Torra {12} H-FS (Hesitant fuzzy sets) are described as a function that generates a set of membership values for each domain element.

Various extension of FSs have been discussed based on their need and importance. Some important results regarding the operation of I-FSs has been obtained. As we know different distance measures have been discussed by numerous researchers for different types of FSs. These distance measurements undoubtedly meet the metric's requirements, and the normalized Euclidean distance has certain desirable geometric characteristics. Yet it might not fit as well in practice. For instance, consider three I-FS J, K and L in the equation $\{X = x_1\}$, where J = (1, 0, 0), K = (0, 1, 0), and L = (0, 0, 1). If we interpret using the ten-person deciding model, J = (1, 0, 0) represents ten people who all are in favor of a candidate; K = (0, 1, 0) denotes ten people who all are against him; and L = (0, 0, 1) denotes ten people who all hesitate. So, it makes sense for us to assume that J and L differ less from one another than J and K do. But, for the above-described Euclidean distance, the distance between J and L is nearly identical to the distance between J and K, which does not seem to make sense to us as a result, we offer a broader definition of the distance between I-FSs. in this study based on the definition of similarity measure provided by Li and Cheng [2] our offered distance was proved more reasonable than Li and cheng.

The remaining part of the paper is organized as follows: Preliminaries and fundamental ideas are contained in Section 2. Extension of FSs is specified in Section 3 in terms of their politeness. Section 4 contains properties of I-FSs, and theorem proofs are found in Section 5. A distance measure between I-FS is introduced in Section 6, including new distance measure with a numerical example, and Section 7 contains a conclusion.

2. Preliminaries and Basic Concepts

2.1 Definition

A Fuzzy Set (FS) {15}E in S is an ordered pair set if $\mathscr{s}$ is group of elements denoted generally by $E = \{(\mathscr{s}, \mu_E(\mathscr{s})) \mid \mathscr{s} \in S\}$, where $\mu_E(\mathscr{s})$ is called membership function(M-F) and its value lies in closed interval [0,1].

2.2 Definition

T2FS, {8} An extension of ordinary FS that is T1FS and is characterized by Type-2 M-F $\mu_Z(\mathscr{s}, u)$. Let S be a fixed universe a T2FS $z \subseteq S$ which is interpreted mathematically as $Z = (\mathscr{s}, u, \mu_Z(\mathscr{s}, u)) \mid \mathscr{s} \in S, u \in j_{\mathscr{s}} \subseteq [0,1]$. in which $0 \le \mu_Z(\mathscr{s}, u) \le 1$ It can also be written as

$$Z = \int_{\mathscr{s} \in S} \mu_Z(\mathscr{s})/\mathscr{s} \mid \mathscr{s} \in S, u \in j_{\mathscr{s}} \subseteq [0,1] = \int_{\mathscr{s} \in S} \left[\iint_{u \in j_{\mathscr{s}}} (g_{\mathscr{s}}(u)/u) \right] / \mathscr{s}$$

where $\mu_Z(\mathscr{s}) = \int_{u \in j_{\mathscr{s}}} (g_{\mathscr{s}}(u)/u)$ *is the M-G,* $g_{\mathscr{s}}(u) = \mu_Z(\mathscr{s}, u)$ named as secondary M-F where μ_Z is primary M-F of Z and j_s is called Primary M-F of $\mathscr{s}$.

2.3 Definition

FOU (Footprint of Uncertainty) {10} for T2FS we are having 3-D structure which becomes very difficult for calculation, so we take the base of 3rd dimension to calculate the values which is called FOU. It can be defined as the union of all Primary M-Fs that is $FOU(Z) = \cup_{\mathscr{s} \in S} (j_{\mathscr{s}})$.

3. Extension of Fuzzy sets

A= $\begin{bmatrix} \mathscr{s}_{11} & \cdots & \mathscr{s}_{1n} \\ \vdots & \ddots & \vdots \\ \mathscr{s}_{m1} & \cdots & \mathscr{s}_{mn} \end{bmatrix}$ where $\mathscr{s}_{ij}$ represents evaluation of alternatives A_i under criteria k_j.For a decision-making problem we have

A: Objective.

B: Criteria (k_j).

C: Alternatives (A_i).

3.1 Definition

Intuitionistic Fuzzy set, if a person is representing the ratio of $\mathcal{s}_{ij}$ in terms of M-D and the N-MD pair. An object of the following form is what Atanassov {1} defines as an I-FS J in S.

$$M = \{\mathcal{s}, \mu_J(\mathcal{s}), \nu_J(\mathcal{s}) : \mathcal{s} \in S,\}$$

where $\mu_J(\mathcal{s}) \in [0,1]$ *and* $\nu_J(\mathcal{s}) \in [0,1]\}$ is called as M-D and N-MD respectively such that $0 \leq \mu_J(\mathcal{s}) + \nu_J(\mathcal{s}) \leq 1 \forall \mathcal{s} \in S.$

3.2 Definition

Interval valued I-FS; If an expert gives his decision value of $\mathcal{s}_{ij}$ in terms of interval [L.U] {1} introduced IVIF. Let a set S be fixed, an IVIFS J over S Is an object having the form $J = \mathcal{s}, \mu_J^l(\mathcal{s}), \mu_J^u(\mathcal{s}), \nu_J^l(\mathcal{s}), \nu_J^u(\mathcal{s})$ where $(\mu_J^l(\mathcal{s}), \mu_J^u(\mathcal{s})) \sqsubset [0,1]$ and $\nu_J^l(\mathcal{s}), \nu_J^u(\mathcal{s}) \sqsubset [0,1]$ under the constraint $\mu_J^u(\mathcal{s}) + \nu_J^u(\mathcal{s}) \leq 1.$

3.3 Definition

Hesitant FS: Tora v {12} extended the concept of I-FS to hesitant FS which permits the M-D a discrete set of [0, 1]. If a person rates the value of $\mathcal{s}_{ij} = 0.5, 0.6, 0.55$ Let P be a reference set, then we describe hesitant FS on S in terms of function h that when applied to S yields a subset of [0,1] $E = \mathcal{s}, h_E(\mathcal{s}); \mathcal{s} \in S$ they consider only agree Nance that is why we feel need of dual hesitant fuzzy set.

3.4 Definition

A dual hesitant FS is a type of FS that is defined using two different functions to determine the M-D and N-MD for every set's element. These functions provide two sets of values, one as M-D and another as N-MD, which can be used to represent the degree of uncertainty or hesitation associated with each element's membership in the set. Given a fixed set P, a dual hesitant FS α on P is interpreted as. $\alpha = \big(p, h(p), g(p)\big); p \in P)$

in which $h(p)$ *and* $g(p)$ are two sets of some values in [0,1] signifying the possible M-D and N-MD of the element $p \in P$ to the set α, respectively, under the constraint $0 \leq \gamma, \theta \leq 1 : 0 \leq \gamma^+ + \theta^+ \leq 1$. Where γ^+ *and* θ^+ denotes the maximum of degree of agree Nance and degree of disagree Nance.

3.5 Definition

P-FS: If someone give rating of s_{ij} as (0.7, 0.4) which is not less than 1 then we use P-FS introduced by {13}. Let S be a universe of discourse (UOD), a P-FS in S is given by $E = (s, h_E(s), g_E(s); s \in S)$ where $h_E, g_E: s \to [0,1]$ are M-D and N-MD with condition $\left(h_E(s)\right)^2 + \left(g_E(s)\right)^2 \leq 1$ for all s in S, the degree of indeterminacy is given by $\gamma_E(s) = \sqrt{1 - \left(h_E(s)\right)^2 - \left(g_E(s)\right)^2}$ For connivance zhang and Xu {13} called $h_E(s), g_E(s)$ a P-F number and is represented as $E = (h_E, g_E)$.

3.6 Definition

Hesitant Pythagorean fuzzy set (HPFS); introduced by {7} defined as

$E = \{\left(s, h(s), g(s)\right)\}; s \in S$ with condition $0 \leq \gamma, \theta \leq 1: 0 \leq (\gamma^+)^2 + (\theta^+)^2 \leq 1 \forall s \in S, \gamma \in h(s), \theta \in g(s)$.

3.7 Definition

Linguistic Pythagorean fuzzy set (LPFS); if someone has to say about linguistic behavior for example beauty, we can't say 70 percent or 80 percent beautiful here we use terms like more beautiful very beautiful etc. LPFS was introduced by {5} is defined as $E = \{S, h_E(s), g_E(s); s \in S\}$ where h_E, g_E represents linguistic M-D and N-MD respectively with condition $(h^2 + g^2 \leq t^2)$.

3.8 Definition

Single valued neutrosophic fuzzy set (SVNFS);{4} In this set we have indeterminacy factor as well and is defined as $E = \{S, h_E(s), g_E(s), i_E(s); s \in S\}$ with condition $h_E, g_E, i_E \in [0,1]$ and $0 \leq h_E + g_E + I_E \leq 3$ for each s in S. Here $h_E(s), g_E(s), i_E(s)$ represents M-D, N-MD, and indeterminacy. If a person says 0.5\% is true, 0.7\% not true and 0.2\% is not sure here not sure part is only taken into consideration in neutrosophic set.

3.9 Definition

Fermatean fuzzy set; when someone provides a pair $\left(r_f(s), s_f(s)\right)$ as the M-D and N-MD like (0.9, 0.6) then the condition of I-FS and P-FS are not satisfied.

$(0.9) + (0.6) > 1, (0.9)^2 + (.6)^2 > 1$, however, it satisfies the condition

$(0.9)^3 + (.6)^3 \leq 1$ So F-FSs are here good to control it introduced by {11}. Let S be the UOD and F be the fermatean set defined as

$F = \{\left(s, r_F(s), s_F(s)\right); s \in S\}$ with condition $0 \leq \left(r_F(s)\right)^3 + \left(s_F(s)\right)^3 \leq 1$.

Also $i_F(\mathcal{s}) = \sqrt[3]{1-\left(r_F(\mathcal{s})\right)^3-\left(s_F(\mathcal{s})\right)^3}$ is identified as degree indeterminacy.

4 Properties of Intuitionistic Fuzzy set operators

Operators of I-FSs {2,3,6,9}, For Every two I-FSs U and V The following Operations and Relations can be defined as Let $\mu_u(\mathcal{s}), \mu_v(\mathcal{s})$be the degree of membership and $\wedge_u(\mathcal{s}), \wedge_v(\mathcal{s})$, are degree of non-membership of fuzzy set U and V respectively than we define

Max Operator as

- $U+V=\{\max\left(\mu_u(\mathcal{s}),\mu_v(\mathcal{s})\right), \min\left(\wedge_u(\mathcal{s}),\wedge_v(\mathcal{s})\right)\}$
- $U*V=\{\min\left(\mu_u(\mathcal{s}),\mu_v(\mathcal{s})\right), \max\left(\wedge_u(\mathcal{s}),\wedge_v(\mathcal{s})\right)\}$

Algebraic Operator

- $U \oplus V = (\mu_u(\mathcal{s})+\mu_v(\mathcal{s})-\mu_u(\mathcal{s})\,\mu_v(\mathcal{s}), \wedge_u(\mathcal{s})\wedge_v(\mathcal{s}))$
- $U \otimes V = (\mu_u(\mathcal{s})\mu_v(\mathcal{s}), \wedge_u(\mathcal{s})+\wedge_v(\mathcal{s})-\wedge_u(\mathcal{s})\wedge_v(\mathcal{s}))$

Einstein Operator

- $U \boxplus V = \frac{\mu_u(\mathcal{s})+\mu_v(\mathcal{s})}{1+\mu_u(\mathcal{s})\mu_v(\mathcal{s})}, \frac{2\wedge_u(\mathcal{s})\wedge_v(\mathcal{s})}{(2-\wedge_u(\mathcal{s}))(2-\wedge_v(\mathcal{s}))+(\wedge_u(\mathcal{s})\wedge_v(\mathcal{s}))}$

- $U \boxtimes V = \frac{2\mu_u(\mathcal{s})\,\mu_v(\mathcal{s})}{(2-\mu_u(\mathcal{s}))(2-\mu_v(\mathcal{s}))+\mu_u(\mathcal{s})\mu_v(\mathcal{s})}, \frac{\wedge_u(\mathcal{s})+\wedge_v(\mathcal{s})}{1+\wedge_u(\mathcal{s})\wedge_v(\mathcal{s})}$

4.1 Proof of theorems

Let U, V and W be three Intuitionistic fuzzy sets, $\mu_u(\mathcal{s}), \mu_v(\mathcal{s})$ *and* $\mu_w(\mathcal{s})$ $\wedge_u(\mathcal{s}), \wedge_v(\mathcal{s})$ *and* $\wedge_w(\mathcal{s})$ be the corresponding membership and non-membership respectively.

Theorem 4.1

$$\mathbf{U} \cup (\mathbf{V} \cap \mathbf{W}) = (\mathbf{U} \cup \mathbf{V}) \cap (\mathbf{U} \cup \mathbf{W})$$

$$LHS = (\mu_u(\mathcal{s}), \wedge_u(\mathcal{s})) \cup (min\left(\mu_v(\mathcal{s}),\mu_w(\mathcal{s})\right), \max\left(\wedge_v(\mathcal{s}),\wedge_w(\mathcal{s})\right)$$

Let $\mu_u(\mathcal{s}) < \mu_v(\mathcal{s}) < \mu_w(\mathcal{s})$ and $\wedge_u(\mathcal{s}) < \wedge_v(\mathcal{s}) < \wedge_w(\mathcal{s})$ then.

$$= (\mu_u(\mathcal{s}), \wedge_v(\mathcal{s})) \cup (\mu_v(\mathcal{s}), \wedge_w(\mathcal{s}))$$

$$= \max\left(\mu_u(\mathcal{s}),\mu_v(\mathcal{s})\right), \min\left(\wedge_u(\mathcal{s}),\wedge_w(\mathcal{s})\right) = \left(\mu_v(\mathcal{s}),, \wedge_u(\mathcal{s})\right)\ldots\ldots\ldots\mathbf{1}$$

$$RHS = (U \cup V) \cap (U \cup V)$$

$$= \{\max(\mu_u(s), \mu_v(s)), \min(\Lambda_u(s), \Lambda_v(s))\} \cap \{\max(\mu_u(s), \mu_w(s)), \min(\Lambda_u(s), \Lambda_w(s))\} .$$

$$= (\mu_v(s), \Lambda_u(s)) \cap (\mu_w(s), \Lambda_u(s))$$

$$= \{\min(\mu_v(s), \mu_w(s)), \max(\Lambda_u(s), \Lambda_u(s))\} = (\mu_v(s), \Lambda_u(s)) \quad \ldots\ldots\ldots \mathbf{2}$$

So, from 1 and 2 us proved intuitionistic fuzzy sets are distributive in nature.

Theorem 4.2

$$\mathbf{U} \cap (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \cap \mathbf{V}) \cup (\mathbf{U} \cap \mathbf{W})$$

Similarly, we can prove the result as proved in theorem 1.

Theorem 4.3

$$\mathbf{U} \otimes \mathbf{V} \sqsubseteq \mathbf{U} \oplus \mathbf{V}$$

$$U \otimes V = \mu_u(s), \mu_v(s), \Lambda_u(s) + \Lambda_v(s) - \Lambda_u(s), \Lambda_v(s)$$

$$U \oplus V = (\mu_u(s) + \mu_v(s) - \mu_u(s)\,\mu_v(s), \Lambda_u(s)\Lambda_v(s))$$

Assume that $\mu_u(s)\mu_v(s) \le \mu_u(s) + \mu_v(s) - \mu_u(s)\,\mu_v(s)$

$$\Rightarrow \mu_u(s)\mu_v(s) - \mu_u(s) - \mu_v(s) + \mu_u(s)\mu_v(s) \le \mathbf{0}$$

$$\Longrightarrow \mu_u(s) + \mu_v(s) - \mu_u(s)\mu_v(s) - \mu_u(s)\mu_v(s) \ge 0$$

$$\Longrightarrow \mu_u(s)(1 - \mu_v(s)) + \mu_v(s)(1 - \mu_u(s)) \ge 0$$

Which is true as $0 \le \mu_u(s) \le 1$ and $0 \le \mu_v(s) \le 1$.

Similarly, $\Lambda_u(s)\Lambda_v(s) \le \Lambda_u(s) + \Lambda_v(s) - \Lambda_u(s)\Lambda_v(s)$

$$\Longrightarrow \Lambda_u(s) + \Lambda_v(s) - \Lambda_u(s)\Lambda_v(s) - \Lambda_u(s)\Lambda_v(s) \ge 0$$

$$\Longrightarrow \Lambda_u(s)(1 - \Lambda_v(s)) + \Lambda_v(s)(1 - \Lambda_u(s)) \ge 0$$

which is true as $0 \le \Lambda_u(s) \le 1$ and $0 \le \Lambda_v(s) \le 1$

Hence $U \otimes V \sqsubseteq U \oplus V$

Theorem 4.4

$$\mathbf{U} \oplus \mathbf{U} \supseteq \mathbf{U}$$

$$\mu_u(\mathscr{s}) + \mu_u(\mathscr{s}) - \mu_u(\mathscr{s})\mu_u(\mathscr{s}), \wedge_u(\mathscr{s})\wedge_u(\mathscr{s})$$

$$\Longrightarrow 2\mu_u(\mathscr{s}) - \left(\mu_u(\mathscr{s})\right)^2, \left(\wedge_u(\mathscr{s})\right)^2$$

$$\Longrightarrow\ 2\mu_u(\mathscr{s}) - \left(\mu_u(\mathscr{s})\right)^2 = \mu_u(\mathscr{s}) + \mu_u(\mathscr{s})\left(1 - \mu_u(\mathscr{s})\right) \geq \mu_u(\mathscr{s})$$

And $\left(\wedge_u(\mathscr{s})\right)^2 \leq \wedge_u(\mathscr{s})$, hence U ⊕ U ⊇ U.

Similarly, we can prove $U \otimes U \subseteq U$.

Theorem 4.5

$$(\mathbf{U})^{(\mathbf{C})^{\mathbf{C}}} = \mathbf{U}$$

$$U = \left(\mu_u(\mathscr{s}), \wedge_u(\mathscr{s})\right)$$

$$U^C = \left(\wedge_u(\mathscr{s}), \mu_u(\mathscr{s})\right)$$

$$(U)^{(C)^C} = \left(\mu_u(\mathscr{s}), \wedge_u(\mathscr{s})\right)$$

Theorem 4.6

$$(\boldsymbol{U} \cup \boldsymbol{V})^{\boldsymbol{C}} = \boldsymbol{U}^{\boldsymbol{C}} \cap \boldsymbol{V}^{\boldsymbol{C}}$$

$$\text{LHS} = (\max\left(\mu_u(\mathscr{s}), \mu_v(\mathscr{s})\right), \min\left(\wedge_u(\mathscr{s}), \wedge_v(\mathscr{s})\right))^{\ C}$$

$$= \min\left(\wedge_u(\mathscr{s}), \wedge_v(\mathscr{s})\right), \max\left(\mu_u(\mathscr{s}), \mu_v(\mathscr{s})\right) \ldots \ldots \ldots \ldots 1$$

$$RHS = U^C \cap V^C = \left(\wedge_u(\mathscr{s}), \mu_u(\mathscr{s})\right) \cap \left(\wedge_v(\mathscr{s}), \mu_v(\mathscr{s})\right)$$

$$= \min\left(\wedge_u(\mathscr{s}), \wedge_v(\mathscr{s})\right), \max\left(\mu_u(\mathscr{s}), \mu_v(\mathscr{s})\right) \ldots \ldots \ldots \ldots 2$$

Hence from 1 and 2 LHS =RHS.

Similarly, we can prove.

Theorem 4.7

$$(\mathbf{U} \cap \mathbf{V})^{\mathbf{C}} = \mathbf{U}^{\mathbf{C}} \cup \mathbf{V}^{\mathbf{C}}$$

Theorem 4.8

$$\mathbf{U} \oplus (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \oplus \mathbf{V}) \cup (\mathbf{U} \oplus \mathbf{W})$$

$$LHS = \left\{\left(\mu_u(\mathscr{s}), \wedge_u(\mathscr{s})\right) \oplus (\mu_v(\mathscr{s}), \wedge_v(\mathscr{s})) \cup (\mu_w(\mathscr{s}), \wedge_w(\mathscr{s}))\right\}$$

$$= \{(\mu_u(s), \Lambda_u(s)) \oplus (\max(\mu_v(s), \mu_w(s)), \min(\Lambda_v(s), \Lambda_w(s)))\}$$

$$= \{(\mu_u(s), \Lambda_u(s)) \oplus (\mu_w(s), \Lambda_v(s))\}$$

$$= \{\mu_u(s) + \mu_w(s) - \mu_u(s)\mu_w(s), \Lambda_u(s)\Lambda_v(s)\} \ldots \ldots .1$$

$$RHS = \{\mu_j(s) + \mu_k(s) - \mu_j(s)\mu_k(s), \Lambda_j(s)\Lambda_k(s)\}$$
$$\cup \{\mu_u(s) + \mu_w(s) - \mu_u(s)\mu_w(s), \Lambda_u(s)\Lambda_w(s)\}$$

Assume that $\mu_u(s) < \mu_v(s) < \mu_w(s)$ and $\Lambda_u(s) < \Lambda_v(s) < \Lambda_w(s)$ then.

$$\max(\mu_u(s) + \mu_v(s) - \mu_u(s)\mu_v(s), \mu_u(s) + \mu_w(s) - \mu_u(s)\mu_w(s)), \min\{(\Lambda_u(s)\Lambda_v(s)), (\Lambda_u(s)\Lambda_w(s))\}$$

$$= \{\mu_u(s) + \mu_w(s) - \mu_u(s)\mu_w(s), \Lambda_u(s)\Lambda_v(s)\} \ldots \ldots \ldots \ldots \ldots 2$$

From 1 and 2 we proved the result.

Similarly, we can prove.

Theorem 4.9

$$U \cup (V \oplus W) = (U \cup V) \oplus (U \cup W)$$

Theorem 4.10

$$\mathbf{U} \boxplus (\mathbf{V} \cup \mathbf{W}) = (\mathbf{U} \boxplus \mathbf{V}) \cup (\mathbf{U} \boxplus \mathbf{W})$$

$$LHS = (\mu_U(s), \Lambda_U(s)) \boxplus (\max(\mu_V(s), \mu_W(s)), \min(\Lambda_V(s), \Lambda_W(s)))$$

Assume that $\mu_U(s) < \mu_V(s) < \mu_W(s)$ and $\Lambda_U(s) < \Lambda_V(s) < \Lambda_W(s)$ then.

$$= (\mu_U(s), \Lambda_U(s)) \boxplus (\mu_W(s), \Lambda_V(s))$$

$$= \frac{\mu_U(s) + \mu_W(s)}{1 + \mu_U(s)\mu_W(s)}, \frac{2\Lambda_U(s)\Lambda_V(s)}{(2 - \Lambda_U(s))(2 - \Lambda_V(s)) + \Lambda_U(s)\Lambda_V(s)} \ldots \ldots \ldots \ldots \ldots \ldots .1$$

$$RHS = (\mu_U(s), \Lambda_U(s)) \boxplus (\mu_V(s), \Lambda_V(s)) \cup (\mu_U(s), \Lambda_U(s)) \boxplus (\mu_W(s), \Lambda_W(s))$$

$$\left\{= \frac{\mu_U(s)+\mu_V(s)}{1+\mu_U(s)\mu_V(s)}, \frac{2\Lambda_U(s)\Lambda_V(s)}{(2-\Lambda_U(s))(2-\Lambda_V(s))+\Lambda_U(s)\Lambda_V(s)}\right\} \cup \left\{\frac{\mu_U(s)+\mu_W(s)}{1+\mu_U(s)\mu_W(s)}, \frac{2\Lambda_U(s)\Lambda_W(s)}{(2-\Lambda_U(s))(2-\Lambda_W(s))+\Lambda_U(s)\Lambda_W(s))}\right\}$$

$$= \max\left(\frac{\mu_U(s) + \mu_V(s)}{1 + \mu_U(s)\mu_V(s)}, \frac{\mu_U(s) + \mu_W(s)}{1 + \mu_U(s)\mu_W(s)}\right),$$

$$\min\left(\frac{2\wedge_U(s)\wedge_V(s)}{(2-\wedge_U(s))(2-\wedge_V(s))+\wedge_U(s)\wedge_V(s)},\frac{2\wedge_U(s)\wedge_W(s)}{(2-\wedge_U(s))(2-\wedge_W(s))+\wedge_U(s)\wedge_W(s)}\right)$$

$Let \mu_U(s) < \mu_V(s) < \mu_W(s)$ and $\wedge_U(s) < \wedge_V(s) < \wedge_W(s)$ then

$$=\frac{\mu_U(s)+\mu_W(s)}{1+\mu_U(s)\mu_W(s)},\frac{2\wedge_U(s)\wedge_V(s)}{(2-\wedge_U(s))(2-\wedge_V(s))+\wedge_U(s)\wedge_V(s)} \ldots\ldots\ldots\ldots\ldots .2$$

From 1 and 2 result is proved

Similarly, we can prove.

Theorem 4.11

$$\mathbf{U} \cup (\mathbf{V} \oplus \mathbf{W}) = (\mathbf{U} \cup \mathbf{V}) \oplus (\mathbf{U} \cup \mathbf{W})$$

5 Distance measure between I-FSs

Since distance measure refers to the distinction between I-FSs, it is conceivable to consider it as a parallel concept to similarity measure. Due to the wide range of real-world applications, they provide, such as pattern identification, machine learning, decision-making, and market forecasting, distance measurements between I-FS, a key notion in fuzzy mathematics, are also attracting a lot of attention. Many distance measurements between I-FSs have been presented and researched in recent years. The following distance measures were put out by szmidt and kacprzyk {17} between J and K:

- **Hamming Distance.**

$$d_H(J,K)=\frac{1}{2}\sum_{j=1}^{n}\{\left|u_j(tj)-u_K(tj)\right|+\left|g_{tj}(u_j)-g_{tj}(u_k)\right|+\left|\phi_j(tj)-\phi_K(tj)\right|$$

- **Normalized Hamming Distance.**

$$d_H(J,K) = \frac{1}{2n}\sum_{j=1}^{n}\{|u_j(tj) - u_K(tj)| + |g_{tj}(u_j) - g_{tj}(u_k)| + |\phi_j(tj) - \phi_K(tj)|$$

- **Euclidean distance.**

$$d_E(J,K) = \{\frac{1}{2}\sum_{j=1}^{n}|u_J(tj) - u_K(tj)|^2 + |g_{tj}(u_J) - g_{tj}(u_k)|^2 + |\phi_J(tj) - \phi_K(tj)|^2\}^{1/2}$$

- **Normalized Euclidean distance.**

$$d_E(J,K) = \{\frac{1}{2n}\sum_{j=1}^{n}|u_J(tj) - u_K(tj)|^2 + |g_{tj}(u_J) - g_{tj}(u_k)|^2 + |\phi_J(tj) - \phi_K(tj)|^2\}^{1/2}$$

These distance measurements undoubtedly meet the metric's requirements, and the normalized Euclidean distance has certain desirable geometric characteristics. Yet it might not fit as well in practice. For instance, consider three I-FS J, K and L in the equation $X = x_1$, where J = (1, 0, 0), K = (0, 1, 0), and L = (0, 0, 1). If we interpret using the ten-person deciding model, J = (1, 0, 0) represents ten people who are in favor of a candidate; K = (0, 1, 0) denotes ten people who all are against him; and L = (0, 0, 1) represents ten people who all hesitate. So, it makes sense for us to assume that J and L differ less from one another than J and K do. But, for the above-described Euclidean distance, the distance between J and L is nearly identical to the distance between J and K, which does not seem to make sense to us. As a result, we provide a more broad definition of distance measure between I-FSs in this study based on the definition of similarity measure provided by Li and Cheng {18} and was proved more reasonable than Li and Cheng.

5.1 New distance measure between I-FSs

For convenience, two IFSs J and K in S are denoted by $J = \{s, u_J(s), v_J(s) \mid s \epsilon S$ and $K = \{s, u_k(s), v_k(s) \mid s \epsilon S$ then we defined new distance for J and K by considering M-F and N-MF.

$$d_1(J,K) = \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_J(s_i) - \mu_K(s_i)| + |\nu_J(s_i) - \nu_K(s_i)|}{4} + \frac{min(|\mu_J(s_i) - \mu_K(s_i)|, |\nu_J(s_i) - \nu_K(s_i)|)}{2}$$

5.1 Definition

A real function $d: F^I(s) \times F^I(s) \rightarrow [0,1]$ is said to be a distance measure, if d satisfies the following axioms:

(A) $0 \leq d(J,K) \leq 1, \forall (J,K) \in F^I(s)$

(B) $d(J,K) = 0, IF\ J = K$

(C) $d(J,K) = d(K,J)$

(D) $If E \subseteq K \subseteq L$ where$J,K,L \in F^I(s)$,then $d(J,L) \geq d(J,K)$ and $d(J,L) \geq d(K,L)$.

Now we will prove the above defined measure is a valid distance measure for I-FS.

$$(A) \quad 0 \leq d_1(J,K) \leq 1$$

Let J and K be two I-FS then we have.

$$|\mu_J(s_i) - \mu_K(s_i)| \geq 0$$

$$|\nu_J(s_i) - \nu_K(s_i)| \geq 0$$

$$\Longrightarrow d_2(J,K) \geq 0$$

then we have $|\mu_J(s_i) - \mu_K(s_i)| \leq 1$ and $|\nu_J(s_i) - \nu_K(s_i)| \leq 1$

$\Rightarrow d_1(J,K) \leq 1$ hence $0 \leq d_1(J,K) \leq 1$.

(B) Holds trivially.

Now we will prove for (C) and (D).

$$(C) \Rightarrow \quad d_1(J,K) = d_1(K,J)$$

we have

$$d_1(J,K) = \frac{1}{n}\sum_{i=1}^{n} \frac{|\mu_J(s_i) - \mu_K(s_i)| + |\nu_J(s_i) - \nu_K(s_i)|}{4} + \frac{min|\mu_J(s_i) - \mu_K(s_i)|, |\nu_J(s_i) - \nu_K(s_i)|}{2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \frac{|\mu_K(s_i) - \mu_J(s_i)| + |\nu_K(s_i) - \nu_J(s_i)|}{4} + \frac{min|\mu_K(s_i) - \mu_J(s_i)|, |\nu_K(s_i) - \nu_J(s_i)|}{2}$$

$$= d_1(K,J)$$

$$\Rightarrow d_1(J,K) = d_1(K,J)$$

Now to prove (D)

$$d_1(J,L) \geq d_1(J,K)$$

it can be easily seen that.

$$|\mu_J(s_i) - \mu_L(s_i)| \geq |\mu_J(s_i) - \mu_K(s_i)|$$

And

$$|\nu_J(s_i) - \nu_L(s_i)| \geq |\nu_J(s_i) - \nu_K(s_i)|$$

so, we have

$$\frac{1}{n}\sum_{i=1}^{n} \frac{|\mu_J(s_i) - \mu_L(s_i)| + |\nu_J(s_i) - \nu_L(s_i)|}{4} + \frac{min|\mu_J(s_i) - \mu_L(s_i)|, |\nu_J(s_i) - \nu_L(s_i)|}{2}$$

$$\geq \frac{1}{n}\sum_{i=1}^{n} \frac{|\mu_J(s_i) - \mu_K(s_i)| + |\nu_J(s_i) - \nu_K(s_i)|}{4} + \frac{min|\mu_J(s_i) - \mu_K(t_i)|, |\nu_J(s_i) - \nu_K(s_i)|}{2}$$

then we get inequality $d_1(J,L) \geq d_1(J,K)$ similarly we can prove $d_1(J,L) \geq d_1(K,L)$ hence satisfies condition (D), so we proved this is a valid distance measure for I-FSs.

5.2 Numerical example for pattern recognition

5.3 Example

Let's consider a pattern recognition problem regarding the classification of industrial materials. Every material is represented by intuitionistic fuzzy sets I_1, I_2, I_3, I_4, I_5 in the feature space $T = \{t_1, t_2, \dots, t_6\}$ (see table 1). We have one unknown industrial material M. Our purpose is to clarify to which class this unknown material belongs. From the data given in table 1 we have following results for $d_1(P,Q)$.

Table 1

	t_1	t_2	t_3	t_4	t_5	t_6
$\mu_{I1}(t)$	0.739	0.033	0.188	0.492	.020	0.739
$\nu_{I1}(t)$	0.125	0.818	0.626	0.358	0.628	0.125
$\mu_{I2}(t)$	0.124	0.030	0.048	0.136	0.019	0.393
$\nu_{I2}(t)$	0.665	0.825	0.800	0.648	0.823	0.653
$\mu_{I3}(t)$	0.449	0.662	1.000	1.000	1.000	1.000
$\nu_{I3}(t)$	0.387	0.298	0.000	0.000	0.000	0.000
$\mu_{I4}(t)$	0.280	0.521	0.470	0.295	0.188	0.735
$\nu_{I4}(t)$	0.715	0.368	0.423	0.658	0.806	0.118
$\mu_{I5}(t)$	0.326	1.000	0.182	0.156	0.049	0.675
$\nu_{I5}(t)$	0.452	0.000	0.725	0.765	0.896	0.263
$\mu_M(t)$	0.629	0.524	0.210	0.218	0.069	0.658
$\nu_M(t)$	0.303	0.356	0.689	0.753	0.876	0.256

$d_1(I_1,M) = 0.199, d_1(I_2,M) = 0.238, d_1(I_3,M) = 0.470, d_1(I_4,M) = 0.147, d_1(I_5,M) = 0.109.$

The material M belongs to I_5. Naturally, this conclusion agrees with Liang and Shi's findings {17}. But our approach is far better as it contains inclusion relation which is failed for many existing measures.

6 Conclusion

In conclusion, the study of FS extensions and I-FSs has significantly expanded our ability to accurately model and analyze real-world systems that exhibit uncertainty and imprecision. By defining various operators and distance metrics, we can manipulate and compare I-FSs, enabling more nuanced and comprehensive analysis. The practical applications of these extensions and metrics are vast, from decision-making processes to image recognition and data compression. With the continued development and implementation of these tools, we can gain deeper insights and make more informed decisions in various fields.

7 References

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