A STUDY OF CALCULATION RULES FOR RINGS

Dr. Niketa Sharma*

Assistant Professor in Mathematics
Hindu Kanya Mahavidyalaya, Jind, Haryana, India

***Email ID**: nikitaakshit@gmail.com*

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Abstract

A ring is a basic structure in abstract algebra and one of the building blocks of mathematics. It's a set that comes with two binary operations, addition and multiplication, that make basic arithmetic more flexible. Polynomials, series, matrices, and functions are all non-numerical objects that can benefit from the application of arithmetic theorems through this generalization. The calculation rules for rings are outlined in this work.

Paper Identification

**Corresponding Author*

Discussion

Assume that $(R; +,\cdot)$ is a commutative ring. Let $a, b, c \in R$.

(i) $If\ a + b = a + c\ then\ b = c.$

(ii) $If\ a + a = a\ then\ a = 0.$

(iii) $-(-a) = a.$

(iv) $0a = 0.$

(v) $-(ab) = (-a)b = a(-b).$

Assume in addition that R has an identity 1 then

(vi) $(-1)a = -a.$

(vii) $If\ a \in R\ has\ a\ multiplicative\ identity\ a - 1\ then\ ab = 0\ implies\ b = 0.$

Proof, Very like the do-it-ideal, adage pursuing proofs you have just observed, or needed to do, with regards to R, or (on account of (i)– (iv)), seen for expansion in a vector space.

Subrings and the Subring Test. [Compare with the Subspace Test from LAI] Let (R; +, •) be a ring and let S be a non-void subset of R. At that point (S; +, •) is a subring of R on the off chance that it is a ring as for the operations it acquires from R.

The Subring Test Let (R; +, •) be a ring and let S ⊆ R. At that point (S; +, •) is a subring of R if (and just if) S is non-vacant and the accompanying hold:

(SR1) $a + b \in S\ for\ any\ a, b \in S;$

(SR2) $a - b \in S\ for\ any\ a, b \in S;$

(SR3) abdominal muscle ∈ S for any $a, b \in S.$

Here (SR1) and (SR3) are only the (bop) conditions we require for S. (A1), (M1) and (D) are acquired from R. The main (∃) sayings are (A2) and (A3). Since S 6= ∅ we can pick some c ∈ S. At that point, by (SR2), $0 = c - c \in S$. By (SR2) once more, $-a = 0 - a \in S$.

Exercise: demonstrate that if S is a subring of R at that point, with an undeniable notation, essentially $0s = 0r$ and $(-a)s = (-a)r$

Precedents

(1) ℤ and ℚ are subrings of R;

(2) R, viewed as quantities of the shape $a + 0i$ for $a \in R$, is a subring of C.

(3) In the polynomial ring [x], the polynomials of even degree from a subring however the polynomials of odd degree don't frame a subring in light of the fact that $x * x = x^2$ isn't of odd degree.

New rings from old: items and capacities. Results of rings Let R1 and R2 be rings. Characterize double operations on R1×R2 facilitate astute:

for $r1, r'1 \in R1$ and $r2, r'2 \in R2$, let

$$(r1, r2) + (r'1, r'2) := (r1 + r1, r2 + r'2),$$

$$(r1, r2) \cdot (r'1, r'2) := (r1 \cdot r'1, r2 \cdot r'2).$$

Consider the ring axioms in two groups:

Type (∀) axioms (no ∃ quantifier): Associativity of addition (A1) and multiplication (M1); Commutativity of addition (A4); distributivity (D). All of these hold because they hold in each coordinate and the operations are defined coordinate wise. Also if multiplication is commutative in R1 and R2 then so is multiplication in R1 × r2.

Type (∃) axioms (containing a ∃ quantifier): existence of zero (A2) and additive inverses (A3). Here we need to exhibit the required elements: (0, 0) serves as zero and $(-r_1, -r_2)$. as the additive inverse of (r_1, r_2). Also, if R1 and R2 are rings with identity, so is $R_1 \times R_2$, with identity $(1, 1)$.

Example: R 2 and more generally R n for n > 2 is a commutative ring with 1 under the coordinate wise operations derived from R.

Rings of functions: [Lecture example] Let R be a ring and X a non-empty set. Denote by RX: = {f: X → R, with sum, f + g, and product, ·, defined point wise:

$$\forall x \in X\,(f + g)(x) = f(x) + g(x),$$

$$\forall x \in X\,(f \cdot g)(x) = f(x) \cdot g(x).$$

Then RX is a ring, which is commutative (has a 1) if R is commutative (has a 1). Many instances of rings can be viewed as subrings of rings of the form RX.

Definition: Let (R, +,.) be a ring. If there is an element 1 ∈ R such that a.1 = 1.a = a for every a ∈ R, then we say that R is a ring with identity (or unit) element. If a.b = b.a for all $a, b \in R$, then we say that R is a commutative ring. A Boolean ring is a ring which satisfies the idempotent law (a.a = a for all a).

Notation: We write ℤ to denote the set of all integers, 2ℤ to denote the set of all even integers, ℚ to denote the set of all rational numbers, ℝ to denote the set of all real numbers and ℤn to denote the set of all integers modulo n.

Examples: (ℤ, +, .) is a commutative ring with identity. (2ℤ, +, .) is a commutative ring without identity. (ℚ, +, .) is a commutative ring with identity. (ℤn, +, .) is a commutative ring with identity.

A commutative ring is said to be an integral domain if it has no ℤero divisors. A ring R is said to be a division ring if (R* , .) is a group where R * = R – {0}.

Definition: An algebraic system (F, +, .) where F is a non-empty set and +, . are binary operations, is said to be a field if it satisfies the following three conditions:

i. (F, +) is an Abelian group;
ii. (F* , .) is a commutative group where F* = F - {0}; and
iii. a(b + c) = ab + ac, (a + b)c = ac + bc for all a, b, c in F.

In other words, a field is a commutative division ring.

Example: Let F be the field of real numbers and let R be the set of all 2 x 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are any real numbers. We define $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{matrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{matrix}$ It is easy to verify that ℝ forms an Abelian group under addition with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ acting as the ℤero element, and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is the additive inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We define the multiplication in ℝ by. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}$.

The element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts as the multiplicative identity element. It is clear that ℝ forms a ring.

Since$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have that ℝ is not an integral domain.

Since$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have that ℝ is not a commutative ring.

Example: (ℤ6, +, .) is a commutative ring but not a division ring.

Definition: Let (ℝ, +, .), (ℝ1 , +, .) be two rings. A mapping ϕ: ℝ → ℝ1 is said to be a homomorphism (or a ring-homomorphism) if it satisfies the following two conditions:

- $\phi\,(a\ +\ b)\ =\ \phi\,(a)\ +\ \phi\,(b);\ and$
- $\phi\,(ab)\ =\ \phi\,(a)\,\phi\,(b)\ for\ all\ a, b\ \in\ R.$

Note: For any two rings R and R1 , if we define ϕ: R → R1 by φ(x) = 0 for all x ∈ R, then ϕ is a homomorphism. This ϕ is called the ℤero-homomorphism.

Definition: Let ϕ: ℝ → ℝ1 be a homomorphism. Then the set {x ∈ ℝ / ϕ (x) = 0} is called the kernal of φ and is denoted by ker ϕ or I(ϕ). A homomorphism ϕ: R → R1 is said to be an isomorphism if ϕ is one-one and onto. Two rings ℝ and ℝ1 are said to be isomorphic if there exist an isomorphism ϕ: ℝ → ℝ1 .

Result: If ϕ is homomorphism from a ring ℝ to ℝ1 , then ϕ(0) = 0 and ϕ(-a) = -ϕ(a) for all a ∈ R.

Definition: Let R be a ring and φ ≠ I ⊆ R. Then

I is said to be a left ideal of R if I is a subgroup of (R, +) and ra ∈ I for every r ∈ R, a ∈ I.

I is said to be a right ideal of R if I is a subgroup of (R, +) and ar ∈I for every r ∈ R, a ∈ I.

I is said to be an ideal (or two sided ideal) of R if I is both left and right ideal. For an ideal I of R, the quotient ring of R with respect to I is denoted by R/I.

An ideal I of R is said to be minimal if it is minimal in the set of all non-ℤero ideals of R.

Remark: (i) Let X ⊆ R. The intersection of all ideals of R containing X is called the ideal generated by X. It is denoted by ‹X›.

If X = {a}, then we write ‹a› for ‹X›.

For an element a ∈ R, the ideal is called the principal ideal generated by a.

It is easy to verify that for a ∈ R the following is true:

$$\langle a\rangle = \{ra + as + (\sum_{i=1}^{k} r_i a s_i) + na \,/\, r_i, s_i \in R, n, k \in \mathbb{Z}\}.$$

Example: If ϕ: R → R1 is a homomorphism, then ker ϕ is an ideal of R.

Lemma: Let f: R → R1 be a ring homomorphism.

If J is an ideal of R, then f(J) = {f(x) / x ∈ J} is an ideal of R1 .

If J1, J2 are ideals of R such that f(J1) and f(J2) are equal, and ker f ⊆ J2, then J1 ⊆ J2 and

If W1 is an ideal of R1 then f-1(W1) = {x ∈ R / f(x) ∈ W1 } is an ideal of R.

Definition: Let R be a ring and S ⊆ R. If S is a ring in its own rights with respect to the operations on R, then S is called a subring of R (equivalently, if (S, +) is a subgroup of (R, +) and ab ∈ S for all a, b ∈ S, then S is called a subring of R).

Example: Let R* = $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} I a, b\varepsilon\, R\right\}$ where R is the set of all real numbers. Then R* is a subring of the ring of 2 x 2 matrices over the field of real numbers, and I = $\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} I, b\varepsilon\, R\right\}$ is an ideal of R*.

If we define ϕ: R* → ℝ by ϕ $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ = a, then ϕ is a homomorphism and I = ker ϕ. So R* /I ≅ (image of ϕ) = ℝ.

Example: Let R be any ring with 1 and a ∈ R. Then Ra = {xa / x ∈ R} is a left ideal and aR = {ar / r ∈ R}is a right ideal of R. If R is a commutative ring, then Ra = aR is an ideal of R.

Example: If U, V are ideals of R, then the two sets U + V = {a + b / a ∈ U, b ∈ V}; and

$$UV = \{ \sum_{i=1}^{n} a_i b_i \ / a_i \in U, b_i \in V \text{ and n is a positi}$$

are ideals of R.

If A is a right ideal and B is a left ideal of R, then A ∩ B need not be a left (or right) ideal of R. We observe this by a suitable example. Let R1 = M2(R) = the ring of all 2 x 2 matrices over the set of real numbers ℝ.

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$A = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} /a, b\varepsilon\ R\right\}$ *and* $B = \left\{\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} /c, d\varepsilon\ R\right\}$.

Then A is a right ideal of R and B is a left ideal of R.

Now A ∩ B= $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} /a\ \varepsilon\ R\right\}$. Let a ∈ R, a ≠ 0.

Since $\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} . \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3a & 0 \\ a & 0 \end{pmatrix} \notin A \cap B$, we conclude that A ∩ B cannot be a left ideal.

Since $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3a & 4a \\ a & 0 \end{pmatrix} \notin A \cap B$, we conclude that A ∩ B cannot be a right ideal.

Therefore A ∩ B is neither a left ideal nor a right ideal of R.

Definitions: (i) An ideal I of R is said to be a maximal ideal if it is maximal in the set of all proper ideals of R.

i. A ring R is called simple if it has exactly two ideals (that is, (0) is the maximal ideal of R).
ii. A ring R is said to be primitive if (0) is the largest amongst the ideals of R contained in the maximal right ideal.
iii. An ideal P of R is said to be prime if A, B are two ideals of R, and AB ⊆ P ⇒ A ⊆ P or B ⊆ P (equivalently, a, b ∈ R and aRb ⊆ P ⇒ a ∈ P or b ∈ P).

We allude to Herstein, Hungerford, Lambek, or Satyanarayana and Syam Prasad for essential definitions and results which are not said here. In this thesis we utilize the expression "Ring" to mean a cooperative ring.

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