# NILRADICAL OF A RING

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**Abstract**

*In algebra, the nilradical of a commutative ring is the ideal consisting of the nilpotent elements:* $\mathfrak{N}_R = \{f \in R | f^m = 0\ for\ some\ m \in \mathbb{Z}_{>0}\}$. *It is thus the radical of the zero ideal. If the nilradical is the zero ideal, the ring is called a reduced ring. The nilradical of a commutative ring is the intersection of all prime ideals. In the non-commutative ring case the same definition does not always work. This has resulted in several radicals generalizing the commutative case in distinct ways; see the article Radical of a ring for more on this. The nilradical of a Lie algebra is similarly defined for Lie algebras.*

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## INTRODUCTION

### LOWER AND UPPER NIL RADICAL OF A RING

**Definition:** An ideal P of a ring R is said to be prime ideal if $P \neq R$ and for ideals$I, J\ of\ P$, $IJ \subseteq P$ implies either $\subseteq P\ or\ J \subseteq P$. Equivalently, for any $a, b \in P$, $aRb \subseteq P$ implies either $a \in P$ or $b \in P$.

**Definition:** A nonempty subset S of a ring R is said to be an $m$-system, if for any $a, b \in S$ there exists $r \in R$ such that $arb \in S$

**Definition:** For an ideal $I$ of a ring $R$, the subset $\{s \in R$ : every m-system containing s meets 7} of R is called the radical of the ideal $I$ and is denoted by $\sqrt{I}$. We note that $\sqrt{I} \subseteq \{s \in R : s^n \in 7$ for some$n \geq 1\}$. Further, equality holds if R is a commutative ring. Further $\sqrt{I}$ is the intersection of all prime ideals of R containing $I$ and hence $\sqrt{I}$ is an ideal of $R$

**Definition:** For a ring R, we define Nil*($R$) =$\sqrt{0}$). This is called lower nilradical of R. Clearly, Nil*® is the smallest prime ideal of R and Nil* $(R) \subseteq J(R)$.

**Definition:** The sum of all nil ideals of a ring R is the largest nil ideal of R and is called upper nil radical of R and denoted by Nil*( $R$).

**SEMI PRIMARY RING:**

A ring R is said to be semi primary if $J(R)$ is nilpotent and $R/J(R)$ is semisimple.

**T-Nilpotent:** An ideal $N$ AT of a ring $R$ is said to be left (resp. right) T-nilpotent ("T" for transfinite) if given any sequence $\{a_i\}$ of elements in N, there exists an $n$ such that $a_1 \ldots a_n = 0$ (resp $a_n \ldots . a_1 = 0$). We note that if $I$ is any ideal of $R$ then $I$ nilpotent ⇒ $I$ is left (right) T-nilpotent => $I$ is nil

**Definition:** An epimorphism is said to be a projective cover of an $R$-module $M$ if $ker(f)$ is a projective module.

A ring R is said to be a left perfect if every R-module has a projective cover and is said to be a left semi perfect if every cyclic $R$ -module has a projective cover.

In the following definition by a regular element, we mean an element of R which is neither a left nor a right zero divisors in $R$.

**LEFT QUOTIENT RING:**

Let R be a ring with regular element. Then a ring Q® is said to be a left quotient ring of R if the following conditions are satisfied.

i) R is a subring of Q®.

ii) Every regular element of R has a two-sided inverse in Q®.

iii) Every element of Q® has the form $c^{-1}a$, where $a, c \in R$ and $c$ is a regular element.

**IDEMPOTENT REFLEXIVE RING:**

A ring R is said to be idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for any $a \in R$ and $e = e^2 \epsilon R$.

Zero Commutative Ring: A ring R is said to be zero commutative (ZC) if for any $a, b \in R, ab = 0$ implies$ba — 0$. We note that reduced rings are ZC.

**DUO RING:**

A ring R is said to be left (resp. right) duo if every left (resp. right) ideal of R is a two sided ideal.

**QUASI DUO RING:**

A ring R is said to be left (resp. right) quasi duo if every maximal left (resp. right) ideal of R is a two sided ideal.

**WEAKLY REGULAR RING:**

A ring R is said to be left (resp. right) weakly regular if $a \in RaRa$ (resp. $a \in aRaR$), for any $a \in R$. We say that a ring R is weakly regular if it is both left and right weakly regular.

Von Neumann regular ring, biregular ring and in particular, any simple ring with identity is weakly regular. But there are weakly regular rings which are neither von Neumann regular nor biregular.

**MATRIX RINGS:**

**Definition:** Let R be a ring and n be a positive integer. Let $Mn(R)$ or Rn denote the set of all n x n matrices with entries from the ring R. i.e. $Mn(R) = \{A$ $=$ ((%•)) : chj £ Rfor every 1 < i,j < n$\}$. It is easy to verify that $Mn(R)$ with usual addition and multiplication of matrices forms a ring. This ring $Mn(R)$ is called the ring of n*n matrices over the ring R.

It is easy to check that $Mn(R)$ is both a left and right R-module.

**Definition:** Let R be a ring. A matrix $A = ((\%)) € Mn(R)$ is said to be lower (resp. upper) tringular if all the entries above (resp. below) the main diagonal are zero, i.e aij = 0 for i < j (resp. i> j).

Let R be a ring and n G N be an integer. Let LTn(R) (resp. UTn(R)) denotes the set of all n x n lower (resp. upper) triangular matrices over R. Then LTn(R) (resp. UTn(R)) is a subring of Mn(R). This subring is called upper (resp. lower) triangular matrix ring.

**Remark:** For 1 < i,j < n, define the element $Etj$ £ $Mn(R)$ by

$$(Eij)rs = \begin{cases} 1 & if\ (r,s) = (i,j) \\ 0 & if\ (r,s) \neq (i,j) \end{cases}$$

These $n^2$ elements of $Mn(R)$ satisfy following identities.

1) $E_{ij}\ E_{jk} = E_{\ ik}$ for every $i,j,k\ \in \{1\ldots n\}$.
2) $E_{ij}E_{ki} = 0$ if $j \neq k$, for every $i,j,k \in \{1\ \ldots\ n\}$.
3) $\sum_{i=1}^{n} E_{ii} = I_n$
4) $If\ A = ((aij)) \in M_n\ (R)$, then $A = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, E_{ij}$

Hence $\{E_{ij}\}_{1\le i,j\le n}$ generate $Mn(R)$ both as a left as well as right Rmodule. $E_{ij}'s$ s are called the matrix units of the matrix ring $Mn(R)$. Also $\{E_{ij} : E_{ij} = 0\ for\ i < j\}$ $(resp. \{E_{ij} : E_{ij} = 0\ for\ i > j\})$ generate $LTn(R)$ (resp. $UTn(R)$) both as a left as well as right $R$ module.

For the sake of convenience, let us recall the following definitions and results from commutative ring theory. The rings considered in this thesis are commutative with identity which is not integral domains. Let R be such a ring. Let M be a unitary Module; By the set of zero-divisors of M as an R-module denoted by$Z_R(M)$, we mean$Z_R(M) = \{r \in R | rm = 0\ for\ some\ m \in M, m \neq 0\}$. We denote $Z_R(R)$ simply by $Z_R$. Recall that a prime ideal P of R is said to be a maximal N-prime of an ideal I of R, if P is maximal with respect to the property of being contained in $Z_R(R/I)$. Note that $S = R \backslash Z(R)$ is a saturated multiplicatively closed subset of R. Observe that if $x \in Z(R)$, then $Rx \cap S = \emptyset$.

**CONCLUSION**

The nilradical of a commutative ring is the set of all nilpotent elements in the ring, or equivalently the radical of the zero ideal. This is an ideal because the sum of any two nilpotent elements is nilpotent (by the binomial formula), and the product of any element with a nilpotent element is nilpotent (by commutativity). It can also be characterized as the intersection of all the prime ideals of the ring (in fact, it is the intersection of all minimal prime ideals).

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