# FUZZY SET AND FUZZY REAL NUMBERS TO DEFINE THE CLASSES OF FUZZY SEQUENCE SPACES $l_M( X, \bar{\lambda}, \bar{p})$ AND $l_M( X, \bar{\lambda}, \bar{p}, L)$ BY USING THE ORLICZ FUNCTION

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**Accepted**: 07.02.2023 **Published**: 01.03.2023

**Keywords**: Fuzzy Set, Fuzzy Real Numbers, Orlicz Function, Paranorm.

## Abstract

*In this paper, we use the fuzzy set and fuzzy real numbers to define the classes of fuzzy sequence spaces $l_M( X, \bar{\lambda}, \bar{p})$ and $l_M( X, \bar{\lambda}, \bar{p}, L)$ by using the Orlicz function. Also, we explore some linear topological structures of the spaces. Further, we introduce a paranorm to study some properties of the spaces. Finally, we define a class $\bar{l}_M( X, \bar{\lambda}, \bar{p}, L)$, and using the concept of $\Delta_2 -$ condition, to show the relation between $l_M( X, \bar{\lambda}, \bar{p}, L)$ and $\bar{l}_M( X, \bar{\lambda}, \bar{p}, L)$.*

**Paper Identification**

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## 1. Introduction

Before the introduction of Fuzzy Logic, mathematics was limited only to two conclusions that are true and false (1 & 0) only. But in 1965, Zadeh [1] was the first to establish the concept of fuzzy set and fuzzy set operations. After that several authors have studied various branches of its theory and applications and an enormous number of authors have employed the fuzzy set and fuzzy real numbers in various sequence spaces. Motloka [ 2] defined the boundedness and convergence of a fuzzy sequence and demonstrated that any convergent sequence of fuzzy numbers is bounded. Similarly, Nanda [ 3] defined a new metric to show that a space of a convergent and bounded sequence of Fuzzy real numbers is complete. Later on, Et. M. Savas and Altinok H. [4] proposed certain classes of fuzzy number sequences, examined them, and analyzed some of their properties such as completeness, solidity, symmetry, convergence free, and also looked at various inclusion relations that were pertinent to these classes. Kim, J.M., and et al. [ 10] established fuzzy norms for the novel idea of a fuzzy normed space and investigated how to express a dual space of sequences. Furthermore, the systematic investigation of fuzzy normed linear spaces with various features is discussed [ 11, 12]. In 2021, Paudel and Pahari [7] used the concept of fuzzy to study a few topological structures in fuzzy metric space. Also, in 2022, Paudel and et al.[8] studied the topological structure of P- bounded variation of difference sequence space and introduced the generalized form of the P- bounded variation of fuzzy real numbers.

In this paper, we study the fuzzy sequence spaces defined by Orlicz function with different properties of the spaces. The idea of the Orlicz space was first put forth by Wladyslaw in 1932. The Orlicz function concept is then used by Lindenstrauss and Trzafriri [ 5] to construct the sequence space as follows $l_M = \{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < \infty for\ some\ \rho > 0\}$. The space $l_M$ with the norm $\|x\|$ defined by $\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < 1\right\}$ becomes a Banach Space and it is called Orlicz sequence space

A function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing, and convex with the properties that $M(0) = 0,\ M(t) > 0$ and $M(t) \to 0$ as $t \to \infty$ is called an Orlicz function.

The Orlicz function $M$ is said to be convex if $M(\alpha t_1 + (1-\alpha)t_2) \le \alpha M(t_1) + (1-\alpha)M(t_2)$.

Using the concept of Orlicz function Kuldip, Ayhan and Sonali [ 6] introduced some ideal convergent sequence spaces of fuzzy numbers, examined applications of infinite matrices and λ-convergence of α order in an effort to examine some of the algebraic and topological aspects of these spaces. In 2015, Sarma [9] created a few fuzzy sequence spaces derived from the Orlicz function, investigated their various characteristics, and established a few inclusion properties.

## 2. **Definition and Preliminaries**:

Let $F$ be the universe of discourse and $X \subseteq F$. Then $X$ is said to be fuzzy set in $F$ if $X$ is the collection of order pair $(x, \mu_X)$ where, $\mu_X : F \to [0,1]$. So that the fuzzy set $X$ in $F$ is defined as $X = \{(x, \mu_X(x)) : x \in F$ and $\mu_X : F \to [0,1]\}$. The function $\mu_X$ is called the membership function and $\mu_X(x)$ is called the degree of the element $x$.

Here, $\mu_X(x) = 0$ means $x$ *is* not included in $X$, and $\mu_x(x) = 1$ means $x$ is fully included in X.

A fuzzy real number X is a fuzzy set, or a mapping between each real number ($\mathbb{R}$) and its membership value X($t$), where X : $\mathbb{R} \to I = [\,0, 1]$ such that

The fuzzy number X is

i. normal if there exists $t \in \mathbb{R}$ such that X(t) = 1

ii. convex if for $t, s \in \mathbb{R}$ and $0 \le \theta \le 1$, X ($\theta$ t + ( 1 – $\theta$) s) ≥ min { X(t) , X(s) }

iii. X is upper semi-continuous if for$\varepsilon > 0, X^{-1}([0, a + \varepsilon))$, for all $a \in I$, is open in the usual topology of $\mathbb{R}$.

The $\alpha$- level set on X is denoted by $X^\alpha$ and defined by $X^\alpha = \{\, t \,\epsilon\, \mathbb{R}\colon X(t) \ge \alpha \,\}$.

The collection of all fuzzy numbers with membership values greater than zero is referred to as support of fuzzy a number.

Assume that $\mathbb{R}(I)$ represents the collection of all fuzzy numbers with upper semi-continuity and compact support. In other words, $X \in \mathbb{R}$(I) then for any $\alpha\epsilon[0, 1]$,

$$X^\alpha = \begin{cases} t : X(t) \ge \alpha \; for \alpha \epsilon \; (\,0, 1] \\ t : X(t) > \alpha \; for \alpha = 0 \end{cases}$$

Suppose $X, Y \in \mathbb{R}(I)$, then the addition $[X + Y]^\alpha$and $(aX)^\alpha$ for $\alpha\epsilon[0, 1]$ is defined as $[X + Y]^\alpha = X^\alpha + Y^\alpha$ and $(aX)^\alpha = a\, X^\alpha$

Every real number r $\in \mathbb{R}$ can be expressed as a fuzzy real number $\bar{r}$ as follow:

$$\bar{r}(t) = \begin{cases} 1 \; for \; t = r \\ 0 \;\; otherwise \end{cases}$$

Now, let us consider a relation $\bar{\rho}$ : $\mathbb{R}(I$ x $\mathbb{R}(I) \to \mathbb{R}^*$ defined by

$$\bar{\rho}(X\,, Y) = \underset{0 \le \alpha \le 1}{Sup}\, d(X^\alpha\,, Y^\alpha)$$

where, $\mathbb{R}^* = \mathbb{R} \cup \{0\}$ and $d$ is a metric defined on the set $G$ of closed and bounded intervals X $= [\,x_1\,, x_2]$on the real line $\mathbb{R}$ defined as

$d(\,A\,, B\,) \;=\; max\,\{\,|x_1 - y_1|, |x_2 - y_2|\}$ for X = $[\,x_1, x_2]$ and $Y = [y_1\,, y_2] \in G$ .

Then, $\bar{\rho}$ defines a metric on $\mathbb{R}(I)$and $(\mathbb{R}(I), \bar{\rho}\,)$ is a complete metric space.

The metric $\bar{\rho}$ defined on $\mathbb{R}(I)$ is said to be translation invariant if

$$\bar{\rho}(X + Y, Z + Y) = \bar{\rho}(\,X, Z)$$

Consider a binary operation $*$: [0, 1] × [0, 1] ⟶ [0, 1] satisfying the following conditions

(i) $1 * x = x$,

(ii) $x * y = y * x$

(iii) $x * y \ge p * q$ whenever $x \ge p$ and $y \ge q$,

(iv) $(x * y) * z) = x * (y * z)$ , for all $x, y\,, z, p\,, q \in [\,0, 1]$

Then the binary operation $*$ is called continuous triangular norm (continuous t-norm) [ **15**]

Example: Let, $a * b$ = min ($a$, $b$), then $*$ is a continuous t-norm.

Let $F$ be a vector space and $X$ is a fuzzy set on $F \times (0, \infty)$ and $*$ is a continuous t-norm then the 3-tuple $(F, X, *)$ is called fuzzy normed space[14] if

i. $X(x, t) > 0$

ii. $X(x, t) = 1$ if and only if $x = 0$

iii. $X(\alpha x, t) = X(x, \frac{t}{|\alpha|})$ for $\alpha \neq 0$.

iv. $X(x, t) * X(y, s) \leq X(x + y, t + s)$

v. $X(x, .): (0, \infty) \to [0, 1]$ is continuous.

vi. $\lim_{t \to \infty} X(x, t) = 1$ and $\lim_{t \to 0} X(x, t) = 0$

A complete fuzzy normed space is called *Fuzzy Banach Space* **.**

Example: Let $(F, \|.\|)$ be a linear normed space and defined a mapping $X(x, t) = \frac{t}{t + \|x\|}$ for all $x \in X$, $t > 0$. Then $(F, X, .)$ is a fuzzy normed space.

Let $(F, \|.\|)$ be a normed linear space and define a mapping $X(x, t) = e^{-(\frac{\|x\|}{t})}$ for all $x \in X$ and $t > 0$. Then $(F, X, .)$ is a fuzzy normed space.

**Paranormed Space [ 9]**: Let X be a vector space. A function $\xi : X \to \mathbb{R}$ satisfying the following

i. $\xi(0) = 0$

ii. $\xi(x) \geq 0$ for all $x \in X$.

iii. $\xi(-x) = \xi(x)$ for all $x \in X$.

iv. $\xi(x + y) \leq \xi(x) + \xi(y)$, for all $x, y \in X$

v. if $(a_n)$ is a sequence of scholars with $a_n \to a$ as $n \to \infty$ and $\{x_n\}$ is a sequence of

such that $\xi(x_n - x) \to 0$ as $n \to \infty$ then $\xi(a_n x_n - ax) \to 0$ as $n \to \infty$ is called paranormed and $(X, \xi)$ is paranormed space.

We note that a paranorm $\xi$ with $\xi(x) = 0$ implies $x = 0$ is called total paranorm.

**Bounded fuzzy set**: A fuzzy set $A$ in X is said to be bounded above if there exists a fuzzy number $M$ in $X$ such that $a \leq M$ for all $a \in A$ and $M$ is called upper bound for $A$.

The fuzzy number $M$ is called the supremum of $A$ if $M$ is an upper bound of $A$ and $M \leq W$ for any upper bound $W$ of $A$ and we write $M = \sup_{a \in A} a$.

Also, fuzzy set $A$ in $X$ is said to be bounded below if there exists a fuzzy number $m$ in $X$ such that $b \geq m$ for all $b \in A$ and $m$ is called lower bound for $A$.

The fuzzy number $m$ is called the infimum of $A$ if m is an upper bound above of A and $m \geq W$ for any lower bound $W$ of $A$ and we write $m = \inf_{b \in A} b$.

A sequence of fuzzy numbers $X = (X_k)$ is said to be bounded if the set $\{ X_k : k \in \mathbb{N}\}$ is bounded. A sequence $X = (X_k)$ of fuzzy numbers is said to converge to a fuzzy number $X_o$ and we write $\lim_k X_k = X_o$ if, for every ε > 0, there exists a positive integer $n_o = n(\varepsilon)$ such that $\bar{\rho}(X_k , X_o) < \varepsilon$ for all k $\geq n_o$

**Limit supremum and limit infimum [16 ]:** Let X = $\{ X_k\}$ be a bounded sequence of fuzzy numbers. The limit infimum and supremum of the sequence are defined as

$lim\ inf X_k = \lim_{n\to\infty} \inf_{k\geq n} X_k$ and $\lim\sup X_k = \lim_{n\to\infty} \lim_{k\geq n} X_k$

We note that the limit infimum or limit supremum of the bounded sequence of fuzzy numbers may not exist.

Let ω(F) denote the set of all sequences of fuzzy numbers. Then any subsequence of ω(F) is sequence space and is called fuzzy sequence space.

$\Delta_2$**- condition [1]**: An Orlicz function $M$ is said to satisfy $\Delta_2$- condition for all values of $t$ if there exists a constant $\gamma > 0$ such that $M(2t)) \leq \gamma M(t)$."

We note that an Orlicz function $M$ satisfy the relation $M(kt) \leq kM(t)$ for all $t$ with 0 < k < 1.

A fuzzy numbers is sequence space $\omega(F)$ of said to be solid if $(Y_k) \in \omega(F)$ whenever $|Y_k| \leq |X_k|$for all $k \in \mathbb{N}$ for some $(X_k) \in \omega(F)$.

In this paper we use the inequality $|a + b|^{p_k} \leq D\{|a|^{p_k} + |b|^{p_k}\}$ where $a, b \in \mathbb{R}, 0 < p_k \leq \sup p_k = L$ and $D = \max\{ 1, 2^{L-1}\}$.

## 3. Main Works

Let $\bar{p} = (p_k)$ and $\bar{q} = (q_k)$ be any two sequences of strictly positive real numbers and $\bar{\lambda}\ (\lambda_k)$ be sequence of non-zero real numbers. Let us introduce the following classes of fuzzy sequences as follows:

$$l_M( X , \bar{\lambda}, \bar{p} ) = \left\{X = (X_k):\ X_k \in \omega(F), k \geq 1 \text{ and } \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{\rho}\right) < \infty\ for\ some\ \rho > 0\right\}$$

$$l_M( X , \bar{\lambda}, \bar{p} , L) = \left\{X = (X_k):\ X_k \in \omega(F), k \geq 1 \text{ and } \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k/L}}{\rho}\right) < \infty\ for\ some\ \rho > 0\right\}$$

where, $\sup_k p_k = L$.

Remark: The class $l_M( X , \lambda, \bar{p} , L)$ is a subset of the class $l_M( X , \lambda, \bar{p} )$.

For simplicity, when $p_k = 1$ for all values of $k$ , let us denote l by m and when $\lambda_k = 1$ for all values of $k$ then we write a by b.

For the sake of simplicity, let us indicate $l_M(X, \lambda, \overline{p})$ by $l_M(X, \bar{\lambda})$ when $p_k = 1$=1 for all values of k, and $l_M(X, \bar{\lambda}, \overline{p})$ by $l_M(X, \overline{p})$ when k =1 for all values of k. Also if $p_k = \lambda_k = 1$ then we write $l_M(X)$ for $l_M(X, \bar{\lambda}, \overline{p})$.

**Theorem 3.1:** Let $\bar{\lambda} = (\lambda_k)$ and $\bar{\mu} = (\mu_k)$ be non-zero sequences of real numbers and let $t_k = \left|\frac{\lambda_k}{\mu_k}\right|, k \geq 1$. Then $l_M(X, \bar{\lambda}, \overline{p}) \subseteq l_M(X, \bar{\mu}, \overline{p})$ if and only if $\lim\inf_k t_k > 0$.

**Proof**: Suppose that $\lim\inf_k t_k > 0$ holds and let $X = (X_k) \in l_M(X, \bar{\lambda}, \overline{p})$. Then there exists $\rho > 0$ such that $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty$.

Here, $\lim\inf_k t_k > 0$ so there exists $m$ and a positive integer $K$ such that $m|\mu_k|^{p_k} \leq |\lambda_k|^{p_k}$ for $k \geq K$. Let us choose $\rho_1 > 0$ such that $\rho \leq m\,\rho_1$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{\rho_1}\right) &= \sum_{k=1}^{\infty} M\left(\frac{|\mu_k|^{p_k}\|X_k\|^{p_k}}{\rho_1}\right)\\ &\leq \sum_{k=1}^{\infty} M\left(\frac{|\lambda_k|^{p_k}\|X_k\|^{p_k}}{m\,\rho_1}\right)\\ &= \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{m\rho_1}\right)\\ &\leq \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty\end{aligned}$$

$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{\rho_1}\right) \Longrightarrow X = (X_k) \in l_M(X, \bar{\mu}, \overline{p})$ and hence $l_M(X, \bar{\lambda}, \overline{p}) \subseteq l_M(X, \bar{\mu}, \overline{p})$.

Conversely, suppose that $l_M(X, \bar{\lambda}, \overline{p}) \subseteq l_M(X, \bar{\mu}, \overline{p})$. Then we show that, $\lim\inf_k t_k > 0$. But we assume that $\lim\inf_k t_k = 0$, then we can find a sequence $k(n)$ of integers such that $k(n+1) \geq k(n) \geq 1, n \geq 1$ for which $n^2|\lambda_{k(n)}|^{p_k} \leq |\mu_k|^{p_k}$. Let, $Z \in X$ with $\|Z\| = 1$ and define $X = (X_k)$ by the relation

$$X_k = \begin{cases}\lambda_{k(n)}^{-1}\, n^{-2/p_{k(n)}}\, Z & for\ k = k(n), n \geq 1\\ 0 & otherwise\end{cases}$$

Let, $\rho > 0$. Then $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) = \sum_{k=1}^{\infty} M\left(\frac{\left\|n^{-2/p_{k(n)}}\, Z\right\|^{p_k}}{\rho}\right)$

$$= \sum_{k=1}^{\infty} M\left(\frac{\|Z\|^{p_k}}{n^2\rho}\right)$$

$$= \sum_{k=1}^{\infty} M\left(\frac{1}{n^2\rho}\right)$$

$$\leq M\left(\frac{1}{\rho}\right)\sum_{k=1}^{\infty}\frac{1}{n^2} < \infty \quad \text{( using the convexity of M)}$$

$$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty \Rightarrow X \in l_M\left(X, \bar{\lambda}, \bar{p}\right).$$

Also, $\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{\rho}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|\mu_{k(n)}\ \lambda_{k(n)}^{-1}\ n^{-2/p_{k(n)}}\ z\right\|^{p_k}}{\rho}\right)$

$$= \sum_{k=1}^{\infty} M\left(\frac{\left\|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right\|^{p_k}}{n^2\rho}\right)$$

$$\geq \sum_{k=1}^{\infty} M\left(\frac{1}{\rho}\right) = \infty$$

$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) = \infty \Rightarrow X \notin l_M(X, \bar{\mu}, \bar{p})$. This contradicts that $l_M\left(X, \bar{\lambda}, \bar{p}\right) \subseteq l_M(X, \bar{\mu}, \bar{p})$. So we must have $\lim\inf_k t_k > 0$. Proved

**Theorem 3.2:** $l_M(X, \bar{\mu}, \bar{p}) \subseteq l_M\left(X, \bar{\lambda}, \bar{p}\right)$ if and only if $\lim\sup_k t_k < \infty$.

**Proof:** Suppose that $\lim\sup_k t_k < \infty$ holds and let $X = (X_k) \in l_M(X, \bar{\mu}, \bar{p})$. Then there exists $\rho > 0$ such that $\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{\rho}\right) < \infty$.

Here, $\lim\sup_k t_k > 0$, so there exists $L > 0$ and a positive integer $K$ such that $L\,|\mu_k|^{p_k} > |\lambda_k|^{p_k}$ for $k \geq K$.

Let us choose $\rho_2 > 0$ such that $L\rho \leq \rho_2$. Then,

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho_2}\right) = \sum_{n=1}^{\infty} M\left(\frac{|\lambda_k|^{p_k}\|X_k\|^{p_k}}{\rho_2}\right)$$

$$\leq \sum_{n=1}^{\infty} M\left(\frac{L|\mu_k|^{p_k}\|X_k\|^{p_k}}{\rho_2}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{m\rho_1}\right)$$

$$\leq \sum_{n=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{\rho}\right) < \infty$$

$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_{k\,k} X_k\|^{p_k}}{\rho_1}\right) < \infty. \Rightarrow X = (X_k) \in l_M\left(X, \bar{\lambda}, \bar{p}\right)$ and hence

$\therefore l_M(X, \bar{\mu}, \bar{p}) \subseteq l_M\left(X, \bar{\lambda}, \bar{p}\right)$.

Conversely, suppose that $l_M(X, \bar{\mu}, \bar{p}) \subseteq l_M\left(X, \bar{\lambda}, \bar{p}\right)$. Then we show that, $\lim\sup_k t_k < \infty$. But we assume that $\lim\sup_k t_k = \infty$. Then we can find a sequence $k(n)$ of integers such that $k(n+1) \geq k(n) \geq 1, n \geq 1$ for which $\left|\frac{\lambda_{k(n)}}{\mu_{k(n)}}\right|^{p_k} > n^2$.

Let, $Z \in X$ with $\|Z\| = 1$ and define $X = (X_k)$ by the relation

$$X_k = \begin{cases} \mu_{k(n)}^{-1}\, n^{-2/p_{k(n)}} & for\ k = k(n), n \geq 1 \\ 0 & otherwise \end{cases}$$

Let, $\rho > 0$. Then $\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k X_k\|^{p_k}}{\rho}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\| n^{-2/p_{k(n)}}\ Z\right\|^{p_{k(n)}}}{\rho}\right)$

$$= \sum_{n=1}^{\infty} M\left(\frac{\|Z\|^{p_{k(n)}}}{n^2\rho}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{1}{n^2\rho}\right)$$

$\leq M\left(\frac{1}{\rho}\right)\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$ ( using the convexity of M) $\therefore$ $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty \implies X \in l_M(X, \bar{\mu}, \bar{p})$.

Also, $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|\lambda_{k(n)}\, \mu_{k(n)}^{-1}\, n^{-2/p_{k(n)}}\ Z\right\|^{p_{k(n)}}}{\rho}\right)$

$$= \sum_{n=1}^{\infty} M\left(\frac{\left|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right|^{p_{k(n)}}}{n^2\rho}\right)$$

$$\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\rho}\right) = \infty$$

$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) = \infty \implies X \notin l_M(X, \bar{\lambda}, \bar{p})$.

This contradicts that $l_M(X, \bar{\mu}, \bar{p}) \subseteq l_M(X, \bar{\lambda}, \bar{p})$. So we must have $\lim\sup_k t_k < \infty$. Proved.

**Theorem 3.3:** If $0 < p_k \leq q_k < \infty$ for all but finitely many values of $k$, then $l_M(X, \bar{\lambda}, \bar{p}) \subseteq l_M(X, \bar{\lambda}, \bar{q})$.

**Proof**: Let $X = (X_k) \in l_M(X, \bar{\lambda}, \bar{p})$, then there exists $\rho > 0$ such that $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty$. This relation shows that there exists $K \geq 1$ such that $\|\lambda_k X_k\| < 1$. So that $\|\lambda_k X_k\|^{q_k} \leq \|\lambda_k X_k\|^{p_k}$ for all $k \geq K \implies \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{q_k}}{\rho}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty$. This shows that $X = (X_k) \in l_M(X, \bar{\lambda}, \bar{q})$ and hence then $l_M(X, \bar{\lambda}, \bar{p}) \subseteq l_M(X, \bar{\lambda}, \bar{q})$.

4. **Linear Topological Structure.**

In this section, we will discuss about the linear topological structure of $l_M(X, \bar{\mu}, \overline{p}, L)$.

**Theorem 4.1:** The class $l_M(X, \bar{\mu}, \overline{p})$ forms a linear space.

Proof: Suppose $X = (X_k)$ and $Y = (Y_k)$ be two elements of $l_M(X, \bar{\lambda}, \overline{p})$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that, $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho_1}\right) < \infty$ and $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho_2}\right) < \infty$. Then for any scholar $\alpha$ and $\beta$, let us choose $\rho_3 > 0$ such that

$$2D\rho_1 \max(1, |\alpha|) \le \rho_3 \text{ and } 2D\rho_2 \max(1, |\beta|) \le \rho_3$$

Now, $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k(\alpha X_k+\beta Y_k)\|^{p_k}}{\rho_3}\right) = \sum_{k=1}^{\infty} M\left(\frac{\|\alpha\lambda_k X_k + \beta\lambda Y_k\|^{p_k}}{\rho_3}\right)$

$$\le \sum_{k=1}^{\infty} M\left(\frac{D\|\alpha\lambda_k X_k\|^{p_k} + D\|\beta\lambda_k X_k\|^{p_k}}{\rho_3}\right)$$

(using the inequality $|X+Y|^{p_k} \le D\{|X|^{p_k} + |Y|^{p_k}\}$ ) where, $\sup p_k = L$ and $D = \max\{1, 2^{L-1}\}$.

$$= \sum_{k=1}^{\infty} M\left(\frac{D\,|\alpha|^{p_k}\|\lambda_k X_k\|^{p_k}}{\rho_3} + \frac{D\,|\beta|^{p_k}\|\lambda_k Y_k\|^{p_k}}{\rho_3}\right)$$

$$\le \sum_{k=1}^{\infty} M\left(\frac{1}{2\rho_1}\|\lambda_k X_k\|^{p_k} + \frac{1}{2\rho_2}\|\lambda_k Y_k\|^{p_k}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho_1}\right) + \frac{1}{2}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k Y_k\|^{p_k}}{\rho_2}\right) < \infty$$

$$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\lambda(\alpha X_k+\beta Y_k\|^{p_k}}{\rho_3}\right) < \infty$$

$\Longrightarrow \alpha X_k + \beta Y_k \in l_M(X, \bar{\lambda}, \overline{p})$ for $X = (X_k)$, $Y = (Y_k) \in l_M(X, \bar{\lambda}, \overline{p})$ and for any scalar $\alpha$ and $\beta$. Hence $l_M(X, \bar{\lambda}, \overline{p})$ is linear. Proved.

**Theorem 4.2 :** If $l_M(X, \bar{\lambda}, \overline{p})$ is linear space then $\lim sup_k t_k < \infty$.

Proof : Suppose $l_M(X, \bar{\lambda}, \overline{p})$ is a linear space, but we assume that $\lim sup_k t_k = \infty$. Then there exists a sequence $\{k(n)\}$ of positive integers such that $k(n+1) > k(n) \ge 1, n \ge 1$, for which $p_{k(n)} > n$ for each $n \ge 1$.

Let us choose $Z \in X$ such that $\|Z\| = 1$ and define a sequence $X = (X_k)$ by the relation

$$X_k = \begin{cases} \lambda_{k(n)}^{-1}\, n^{-2/p_{k(n)}}\, Z & for\ k = k(n), n \ge 1 \\ 0 & otherwise \end{cases}$$

Then for $k = k(n), n \ge 1$, we have,

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) = \sum_{k=1}^{\infty} M\left(\frac{\left\|n^{-2/p_{k(n)}} Z\right\|^{p_k}}{\rho}\right)$$

$$= \sum_{k=1}^{\infty} M\left(\frac{1}{n^2\rho}\right)$$

$$\leq M\left(\frac{1}{\rho}\right)\sum_{k=1}^{\infty}\frac{1}{n^2} < \infty s$$

$\therefore X \notin l_M\left(X, \bar{\lambda}, \bar{p}\right)$.

Since, $p_{k(n)} > n$ for $n \geq 1$, so for any $\rho > 0$ and $\alpha = 4$ we have,

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \, \alpha X_k\|^{p_k}}{\rho}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|4n^{-2/p_{k(n)}}Z\right\|^{p_{k(n)}}}{\rho}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{4^{p_{k(n)}}}{n^2\,\rho}\right) \geq \sum_{n=1}^{\infty} M\left(\frac{4^n}{n^2\,\rho}\right) \geq \sum_{n=1}^{\infty} M\left(\frac{1}{\rho}\right) = \infty$$

Thus, $\alpha X_k \notin l_M\left(X, \bar{\lambda}, \bar{p}\right)$, which is a contradicts the linearity of $l_M\left(X, \bar{\lambda}, \bar{p}\right)$. So that $\lim sup_k t_k < \infty$. Proved.

**Theorem 4.3:** **(i)** $l_M\left(X, \bar{\lambda}, \bar{p}\right) \subseteq l_M(X, \bar{p})$ if and only $\lim inf_k|\lambda_k| > 0$,

(ii) $l_M(X, \bar{p}) \subseteq l_M\left(X, \bar{\lambda}, \bar{p}\right)$ if and only $\lim sup_k|\lambda_k|^{p_k} < \infty$ ;

(iii) $l_M(X, \bar{p}) = l_M\left(X, \bar{\lambda}, \bar{p}\right)$ if and only $0 < \lim inf_k|\lambda_k|^{p_k} \leq \lim sup_k|\lambda_k|^{p_k} < \infty$

**Theorem 4.4** : If $\inf_k p_k = l > 0$, then $l_M\left(X, \bar{\lambda}, \bar{p}, L\right)$ forms a paranormed space with respect to $g(\bar{X}) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \, X_k\|^{p_k/L}}{\rho}\right) \leq 1\right\}$.

**Proof**: Let us consider a set $A(\bar{X}) = \left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \, X_k\|^{p_k/L}}{\rho}\right) \leq 1\right\}$. We assume that $\inf_k p_k = l > 0$ and $g(\bar{X}) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \, X_k\|^{p_k/L}}{\rho}\right) \leq 1\right\}$. Then,

i. $$g(0) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \, 0\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

$$= \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|0\|^{p_k/L}}{\rho}\right) \leq 1\right\} = 0$$

ii. $$g(-\bar{X}) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \, (-X_k)\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

$$= \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|-\lambda_k X_k\|^{p_k/L}}{\rho}\right) \le 1\right\}$$

$$= \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k/L}}{\rho}\right) \le 1\right\}$$

$$= g(\bar{X})$$

iii. Let, $X, Y \in l_M(X, \bar{\lambda}, \bar{p}, L)$. Then there exists $\rho_1 > 0$, $\rho_2 > 0$ such that ,

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k/L}}{\rho_1}\right) < \infty \text{ and } \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k/L}}{\rho_2}\right) < \infty$$

Let $\rho_3 = \rho_1 + \rho_2$ where, $\rho_1, \rho_2 \in A(\bar{X})$. Then,

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k(X_k + Y_k\|^{\frac{p_k}{L}}}{\rho_3}\right) = \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k(X_k + Y_k\|^{\frac{p_k}{L}}}{\rho_1 + \rho_2}\right)$$

$$\le \frac{\rho_1}{\rho_1 + \rho_2} \sum_{k=1}^{\infty} M\left(\|\lambda_k X_k\|^{\frac{p_k}{L}}\right) + \frac{\rho_2}{\rho_1 + \rho_2} \sum_{k=1}^{\infty} M \|\lambda_k Y_k\|^{\frac{p_k}{L}}$$

$$\le \frac{\rho_1}{\rho_1 + \rho_2} . 1 + \frac{\rho_2}{\rho_1 + \rho_2} . 1 = 1$$

$$\therefore \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k(X_k + Y_k\|^{\frac{p_k}{L}}}{\rho_3}\right) \le 1$$

This relation shows that $\rho_3 = \rho_1 + \rho_2 \in A(\bar{X})$. So that

$g(X + Y) \le \rho_3 = \rho_1 + \rho_2$ for $\rho_1 \in A(\bar{X})$ and $\rho_2 \in A(\bar{Y})$.

i.e, $g(X + Y) \le \rho_1 + \rho_2$

i.e, $g(X + Y) \le (\bar{X}) + (\bar{Y})$.

Hence the triangle inequality holds.

iv. Suppose $\bar{X}^n = (\bar{X}_k^n)$ be a sequence in $l_M(X, \bar{\lambda}, \bar{p}, L)$ such that $g(\bar{X}^n) \to 0$ as $n \to \infty$ and $(\alpha_n)$ be a sequence of scholar such that $\alpha_n \to \alpha$. Then

$$g(\alpha_n \bar{X}^n) = \inf\left\{ \rho : \sum_{k=1}^{\infty} M\left(\|\lambda_k \alpha_k \bar{X}_k^n\|^{\frac{p_k}{L}}\right) \le 1\right\}$$

$$= \inf\left\{ \rho : \sum_{k=1}^{\infty} M\left(|\alpha_k|^{\frac{p_k}{L}} \|\lambda_k \bar{X}_k^n\|^{\frac{p_k}{L}}\right) \le 1\right\}$$

$$\le \inf\left\{ \rho : \sum_{k=1}^{\infty} M\left(G^{\frac{p_k}{L}} \|\lambda_k \bar{X}_k^n\|^{\frac{p_k}{L}}\right) \le 1\right\}, \text{ where } G = \sup_n |\alpha_n|.$$

Then for $\beta = \max(1, G)$, we have,

$g(\alpha_n \bar{X}^n) \le \inf\left\{ \rho : \sum_{k=1}^{\infty} M\left(\frac{\beta \|\lambda_k \bar{X}_k^n\|^{\frac{p_k}{L}}}{\rho}\right) \le 1\right\}$. Taking $\frac{\rho}{\beta} = \gamma$ then $\gamma > 0$ and

$g(\alpha_n\bar{X}^n) \leq \inf\left\{\rho : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\bar{X}_k^n\|^{\frac{p_k}{L}}}{\gamma}\right) \leq 1\right\}.$

$$=\beta g(\bar{X}^n) \to 0 \text{ as } n \to \infty.$$

$\therefore g(\alpha_n\bar{X}^n) \to 0$ as $n \to \infty$.

Now, let $\alpha_n \to 0$ as $n \to \infty$ then for $0 < \varepsilon < 1$, we can find a positive integer $N$ such that $|\alpha_n| \leq \varepsilon$ for all $n \geq N$ and let $X \in l_M(X, \bar{\lambda}, \bar{p}, L)$. Since $\inf_k p_k = l > 0$, so that $|\alpha_n|^{\frac{p_k}{L}} \leq (\varepsilon)^{\frac{l}{L}}$ for all $n \geq N$.

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_n\lambda_k\bar{X}_k\|^{\frac{p_k}{L}}}{\rho}\right) = \sum_{k=1}^{\infty} M\left(\frac{|\alpha_k|^{\frac{p_k}{L}}\|\lambda_k\bar{X}_k\|^{\frac{p_k}{L}}}{\rho}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\varepsilon^{\frac{l}{L}}\|\lambda_k\bar{X}_k\|^{\frac{p_k}{L}}}{\rho}\right)$$

Thus, if $\rho \in A(\varepsilon^l\bar{X})$ then, $\rho \in A(\alpha_n\bar{X})$ $i.e$ $A(\varepsilon^l\bar{X}) \subseteq A(\alpha_n\bar{X})$.

Taking infimum over such that $\rho' s$, then we get

$$\inf\{\rho : \rho \in A(\alpha_n\bar{X})\} \leq \inf\left\{\rho : \rho \in A\left(\varepsilon^{\frac{l}{L}}\bar{X}\right)\right\} = \varepsilon^{\frac{l}{L}}\inf\{\rho : \rho \in A(\bar{X})\}$$

This shows that, $g(\alpha_n\bar{X}) \leq g(\varepsilon^{\frac{l}{L}}\bar{X})$ for all $n \geq N$. That is $g(\alpha_n\bar{X}) \to 0$ $as$ $n \to \infty$.

Hence, $l_M(X, \bar{\lambda}, \bar{p}, L)$ is paranormed space. This completes the proof.

**Theorem 4.5** : The paranormed space $(l_M(X, \bar{\lambda}, \bar{p}, L), g)$ is complete.

**Proof:** Let $\{\bar{X}_k^i\}$ be a Cauchy sequence in $l_M(X, \bar{\lambda}, \bar{p}, L)$. Let $r > 0$ be a fixed positive real number such that $M(r) \geq 1$. Then for every $\frac{\varepsilon}{r} > 0$, there exists an integer $N \geq 1$, such that $g(\bar{X}_k^i - \bar{X}_k^j) < \frac{\varepsilon}{r}$ for all $i, j \geq N$ . . . (*)

By the definition of paranorm we see that,

$$\sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k\bar{X}_k^i - \lambda_k\bar{X}_k^j\right\|^{\frac{p_k}{L}}}{g(\bar{X}_k^i) - g(\bar{X}_k^j)}\right) \leq 1 \text{ for all } i, j \geq N.$$

So that, $M\left(\frac{\left\|\lambda_k\bar{X}_k^i - \lambda_k\bar{X}_k^j\right\|^{\frac{p_k}{L}}}{g(\bar{X}^i) - g(\bar{X}_k^j)}\right) \leq 1 \leq M(r)$ for all $i, j \geq N$ and for all $k \geq 1$. Since $M$ non decreasing , we have $\frac{\left\|\lambda_k\bar{X}_k^i - \lambda_k\bar{X}_k^j\right\|^{\frac{p_k}{L}}}{g(\bar{X}_k^i) - g(\bar{X}_k^j)} < r.$

Then from (*)we have, $\left\|\lambda_k \bar{X}_k^i - \lambda_k \bar{X}_k^j\right\|^{\frac{p_k}{L}} < \varepsilon$. This shows that $\left\{\bar{X}_k^i\right\}$ is a Cauchy sequence in $\mathbb{R}(I)$ and $\mathbb{R}(I)$ is complete, so there exists $\bar{X}_k$ in $\mathbb{R}(I)$ for all $k \geq 1$ such that $\bar{X}_k^i \longrightarrow \bar{X}_k$ as $i \longrightarrow \infty$. To complete the proof we show that $\bar{X} = (\bar{X}_k) \in l_M(X, \bar{\lambda}, \bar{p}, L)$.

Let us choose $\rho > 0$ such that $g(\bar{X}^i - \bar{X}^j) < \rho < \varepsilon$ for all $i, j \geq N$.

$$\sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k \bar{X}_k^i - \lambda_k \bar{X}_k^j\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k \bar{X}_k^i - \lambda_k \bar{X}_k^j\right\|^{\frac{p_k}{L}}}{g(\bar{X}_k^i) - g(\bar{X}_k^j)}\right) \leq 1 \text{ for all } i, j \geq N$$

Since, $M$ is continuous, taking limit as $j \longrightarrow \infty$ we see that"

$$\sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k \bar{X}_k^i - \lambda_k \bar{X}_k^j\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1 \quad \text{for all } i \geq N.$$

Taking infimum of such $\rho's$, we get

$$\inf \sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k \bar{X}_k^i - \lambda_k \bar{X}_k^j\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1$$

$$\Rightarrow g(\bar{X}^i - \bar{X}) = \inf\left\{\rho: \sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k \bar{X}_k^i - \lambda_k \bar{X}_k^j\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1\right\} \leq \rho < \varepsilon \text{ for all } i \geq N.$$

$$\Rightarrow g(\bar{X}^i - \bar{X}) < \varepsilon \text{ for all } i \geq N.$$

This shows that $\bar{X}^i \longrightarrow \bar{X}$ as $i \longrightarrow \infty$ and clearly, $\bar{X}^i - \bar{X} \in l_M(X, \bar{\lambda}, \bar{p}, L)$, for all $i \geq N$. Hence, $\bar{X}^i$ and $\bar{X}^i - \bar{X}$ are the elements of $l_M(X, \bar{\lambda}, \bar{p}, L)$. So that, $\bar{X} = \bar{X}^i - (\bar{X}^i - \bar{X}) \in l_M(X, \bar{\lambda}, \bar{p}, L)$. That is $\bar{X} \in l_M(X, \bar{\lambda}, \bar{p}, L)$. This completes the proof.

**Theorem 4.6:** The space $l_M(X, \bar{\lambda}, \bar{p}, L)$ is normal.

**Proof:** Let $\bar{X} = (X_k) \in l_M(X, \bar{\lambda}, \bar{p}, L)$. So that $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right) < \infty$ for some $\rho > 0$.

Let $(\alpha_k)$ be sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \geq 1$. Since $M$ is non-decreasing, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k \alpha_k X_k\|^{\frac{p_k}{L}}}{\rho}\right) = \sum_{k=1}^{\infty} M\left(\frac{|\alpha_k|^{\frac{p_k}{L}} \|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right) < \infty$$

Thus, $(\alpha_k X_k) \in l_M(X, \bar{\lambda}, \bar{p}, L)$. So $l_M(X, \bar{\lambda}, \bar{p}, L)$ is normal.

Let us introduce the sub-class $\bar{l}_M(X, \bar{\lambda}, \bar{p}, L)$ of $l_M(X, \bar{\lambda}, \bar{p}, L)$defined as follow

$$\bar{l}_M(X, \bar{\lambda}, \bar{p}, L) = \left\{ X = (X_k):\ X_k \in \omega(F)\ \sum_{k=1}^{\infty} M\left( \frac{\|\lambda_k X_k\|^{p_k/L}}{\rho} \right) < \infty \ for\ every\ \rho > 0 \right\}$$

**Theorem 4.7:** The Orlicz function $M$ satisfies $\Delta_2 - condition$ then $l_M(X, \bar{\lambda}, \bar{p}, L) = \bar{l}_M(X, \bar{\lambda}, \bar{p}, L)$.

**Proof:** Suppose $\bar{X} = (X_k) \in l_M(X, \bar{\lambda}, \bar{p}, L)$ then for some $> 0$ , $\sum_{k=1}^{\infty} M\left( \frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho} \right) < \infty$.

Let us consider an arbitrary $\rho_1 > 0$.

**Case I:** If $\rho \le \rho_1$, then clearly we have, $\sum_{k=1}^{\infty} M\left( \frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho_1} \right) \le \sum_{k=1}^{\infty} M\left( \frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho} \right) < \infty$ and hence we get $\bar{X} = (X_k) \in \bar{l}_M(X, \bar{\lambda}, \bar{p}, L)$.

**Case II :** If $\rho > \rho_1$ , then by using $\Delta_2 - condition$ , we get $\sum_{k=1}^{\infty} M\left( \frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho_1} \right) = \sum_{k=1}^{\infty} M\left( \frac{\frac{\rho}{\rho_1}\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho} \right)$

$$\le K \frac{\rho}{\rho_1} \sum_{k=1}^{\infty} M\left( \frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho} \right) < \infty$$

where, $K$ is the number involved I $\Delta_2 - condition$ of $M$.

Hence $l_M(X, \bar{\lambda}, \bar{p}, L) \subseteq \bar{l}_M(X, \bar{\lambda}, \bar{p}, L)$. And hence $l_M(X, \bar{\lambda}, \bar{p}, L) = \bar{l}_M(X, \bar{\lambda}, \bar{p}, L)$.

**Conclusion:** In this study, we define sequence spaces using the concepts of fuzzy sets and fuzzy real numbers $l_M(X, \bar{\lambda}, \bar{p})$ and $l_M(X, \bar{\lambda}, \bar{p}, L)$ with the help of Orlicz function and studied some linear topological properties of the spaces. Some theorems related to those spaces have been proved. Further, we have defined a paranorm to show that the space $l_M(X, \bar{\lambda}, \bar{p}, L)$ is complete. Moreover, space $l_M(X, \bar{\lambda}, \bar{p}, L)$ is shown as a normal space.

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