RATIONAL TYPE CONTRACTION IN CONE METRIC SPACES VIA FIXED POINT THEOREM

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Abstract

In this manuscript, we create some fixed point theorems for self mapping satisfying rational type contractive conditions in complete cone metric spaces. The delivered results upgrade and federate many existing outcomes on the topic in the literature R. Bhardwaj et al [7]. Some suitable examples verify our results.

Paper Identification

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1. Introduction

The literature of fixed point (FP) for contraction mapping embarked upon by S. Banach, [1] in 1922, which is called the Banach fixed point Theorem (BFPT) or Banach contraction principle (BCP). BCP is stated as follows:

"A single valued contractive type mapping on a complete metric space has a unique FP."

Outcomes of FP (fixed point) treatment with general contractive conditions relating to rational type expression are also interesting. Some notorious results from this track are included (see [2-14]). It has been extended and generalized various types of contraction in different metric spaces.

In 2007, Huang and Zhang [15] developed the concept of the cone metric and obtained some fixed point theorems of contractive mappings. Subsequently, many other authors have generalized and extended the results of Huang and Zhang [15] (see for instance [16-33]).

In 2012, R. Uthaya and G.A. Prabhakar [34] established a common fixed point theorem in cone metric space for rational contraction, which is a generalization of the result of Arshad et al. [35]. S. K. Tiwari et al. [36] also generalized the result of [34]. Pawan Kumar and Ansari [37] used normality congruence to generalize Das and Gupta's [2] result in cone metric space.

Recently, D. Yadav and S. K. Tiwari [38] improved, extended, and widely distributed the results of [37] by the normality of the cone for rational expressions in cone metric spaces.

Therefore, in this work, we discuss, extend, and generalized some fixed point theorems for rational inequality satisfying contrastive type mapping. The delivered results upgrade and federate numerous existing outcomes on the topic in the literature.

2. Preliminaries

First, we recall some standard notations and definitions, which we needed in the sequel.

Definition 2.1 ([15]): Let Ω be a subset of E, where E be a real Banach space and 0 denotes the zero element in E, then Ω is called a cone if and only if:

(i) Ω is a non-empty set closed and $\Omega \neq \{0\}$,

(ii) If ξ_1, ξ_2 are non-negative real numbers and $\upsilon, \xi \in \Omega$, then $\xi_1 \upsilon + \xi_2 \xi \in \Omega$,

(iii) $\upsilon \in \Omega\ and - \upsilon \in \Omega \Rightarrow \upsilon = 0 \Leftrightarrow \Omega \cap (-\Omega) = \{0\}$.

Now specify a partial order $\leq$ on E as for as Ω, whereas provide a cone $\Omega \subset E$ such that $\nu \leq \xi$ if and only if $\xi - \nu \in \Omega$. If $\xi - \nu \in int\, \Omega$ (where int Ω denotes the interior of Ω), then we shall formulate $\nu \ll \xi$. If $int\Omega \neq \emptyset$, then cone Ω is solid. If the least +ve number $\chi > 0$ satisfying the following way

$$\| \nu \| \leq \chi \| \xi \|, \text{ for all } \nu, \xi \in E \text{ and } 0 \leq \nu \leq \xi.$$

The Ω is called normal cone with a normal constant $\boldsymbol{\chi}$.

Even if every growing sequence which is bounded from above is convergent. Then the cone Ω is called regular. That is, if $\{\nu_k\}$ is sequence such that

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_k \leq \cdots \leq \xi.$$

For some, $\xi \in E$, then there is $\nu \in E$ such that $\|\nu_k - \nu\| \to 0 (k \to \infty)$, equivalently, the cone Ω is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 2.1 ([15]): Let $\mathcal{H}$ be a non-empty set. Suppose $\partial: \mathcal{H} \times \mathcal{H} \longrightarrow E$ satisfies

(d_1). $0 < \partial(\nu, \xi)$ for all $\nu, \xi \in \mathcal{H}$ and $(\nu, \xi) = 0$ if and only if $\nu = \xi$;

(d_2). $\partial(\nu, \xi) = \partial(\xi, \nu)$for all $\nu, \xi \in \mathcal{H}$,

(d_3). $\partial(\nu, \xi) \leq \partial(\nu, \zeta) + d(\zeta, \xi)$ for all $\nu, \xi, \zeta \in \mathcal{H}$.

Then the pair $(\mathcal{H}, \partial)$ is called a cone metric space, and ∂ is a cone metric on $\mathcal{H}$.The concept of cone metric space is more general than that of a metric space.

Example 2.2: Let $E = R^2, \Omega = \{(\nu, \xi) \in E: \nu, \xi \geq 0\}$, $\mathcal{H} = \mathcal{R}$ and $\partial: \mathcal{H} \times \mathcal{H} \to E$ defined by $\partial(\nu, \xi) = (|\nu - \xi|, \theta|\nu - \xi|)$, where $\theta \geq 0$ is a constant. Then $(\mathcal{H}, \partial)$ is a cone metric space.

Definition: 2.3([15]): Say $\{\nu_k\}_{k\geq1}$ be a sequence on $(\mathcal{H}, \partial)$, where $(\mathcal{H}, \partial)$ is a cone metric space, $\nu \in \mathcal{H}$. Then,

(1) $\{\nu_k\}_{k\geq1}$ converges to ν whenever for every $\rho \in E$ with $0 \ll \rho$, if there is $\mathcal{N}$ such that $\partial(\nu_k, \nu) \ll \rho$ for all $k \geq \mathcal{N}$. We denote this by $lim_{k\to\infty} \nu_k = \nu$ or $\nu_k \to \nu, (\nu \to \infty)$.

(2) $\{\nu_k\}_{k\geq1}$is said to be a Cauchy sequence if for every $\rho \in E$ with $0 \ll \rho$, if there is $\mathcal{N}$ Such that for all $k. l \geq \mathcal{N}, \partial(\nu_k, \nu_l) \ll \rho$.

(3) Whether every Cauchy sequence is convergent in $\boldsymbol{\mathcal{H}}$.Then $(\mathcal{H}, \partial)$ is called a complete Cone metric space

Lemma 2.4: Let$(\mathcal{H}, \partial)$ be a cone metric space and $\{\nu_k\} \in \mathcal{H}$such that

$$\partial(\nu_{k+1,}\nu_k) \leq \sigma d(\nu_{k,}\nu_{k-1}), \text{ where } 0 \leq \sigma < 1.$$

Then $\{\nu_k\}$ be a Cauchy sequence.

Lemma 2.5([15]) (1). Let $\{\nu_k\}$ be a sequence on cone metric space $(\mathcal{H}, \partial)$ and $\boldsymbol{\chi}$ be a normal constant with normal cone Ω .If $\{\nu_k\}$ converges to ν and $\{\nu_k\}$ converges to ξ, then $\nu = \xi$. That is the limit of $\{\nu_k\}$ is unique.

(2) Let $(\mathcal{H}, \partial)$be a cone metric space, $\{\nu_k\}$ be a sequence in $\boldsymbol{\mathcal{H}}$. If $\{\nu_k\}$ converges toν, then $\{\nu_k\}$ is a Cauchy sequence.

(3) Let $(\mathcal{H}, \partial)$ be a cone metric space, Ω be a normal cone with normal constant $\boldsymbol{\chi}$. Let $\{\nu_k\}$ be a sequence in $\boldsymbol{\mathcal{H}}$. Then $\{\nu_k\}$ is a Cauchy sequence if and only if $d(\nu_k, \nu_l) \to 0$ $(k, l \to \infty)$.

3. Main Result

Theorem 3.1. Consider that, $(H, \leq)$ be a partially ordered set and let there is a cone metric ∂ on $\boldsymbol{\mathcal{H}}$ such that $(\mathcal{H}, \partial)$ be a complete cone metric space and $\boldsymbol{\chi}$ is normal constant with normal cone Ω . Assume that, $\Gamma: \mathcal{H} \to \mathcal{H}$ be a continuous self mapping such that $\nu, \xi \in \mathcal{H}$, $\nu \neq \xi$ there exist $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in [0,1)$ with $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\theta_5 < 2$, satisfies the following condition

$$\partial(\Gamma\nu,\Gamma\xi) \leq \theta_1 \frac{\partial(\nu,\Gamma\nu).\partial(\xi,\Gamma\xi)+\partial(\nu,\Gamma\xi).\partial(\xi,\Gamma\nu)}{\partial(\nu,\Gamma\nu)+\partial(\xi,\Gamma\xi)+\partial(\nu,\Gamma\xi)+\partial(\xi,\Gamma\nu)} + \theta_2 \frac{\partial(\nu,\Gamma\nu).\partial(\xi,\Gamma\nu)+\partial(\xi,\Gamma\xi).\partial(\nu,\Gamma\xi)}{\partial(\nu,\Gamma\nu)+\partial(\xi,\Gamma\nu)+\partial(\xi,\Gamma\xi)+\partial(\nu,\Gamma\xi)}$$

$$+ \theta_3 \frac{\partial(\nu,\Gamma\nu).\partial(\nu,\Gamma\nu)+\partial(\nu,\Gamma\xi).\partial(\xi,\Gamma\nu)}{\partial(\nu,\Gamma\nu)+\partial(\nu,\Gamma\nu)+\partial(\nu,\Gamma\xi)+\partial(\xi,\Gamma\nu)} + \theta_4 \frac{\partial(\nu,\Gamma\nu).\partial(\nu,\Gamma\xi)+\partial(\xi,\Gamma\nu).\partial(\xi,\Gamma\xi)}{\partial(\nu,\Gamma\nu)+\partial(\nu,\Gamma\xi)+\partial(\xi,\Gamma\nu)+\partial(\xi,\Gamma\xi)}$$

$$+ \theta_5\partial(\nu,\xi). \qquad (3.1.1)$$

For $\nu_0 \in \mathcal{H}$, suppose $\{\nu_k\}_{k=0}^{\infty} \subset \mathcal{H}$ defined by $\nu_{k+1} = \Gamma\nu_k, k = 0,1,2 ....$ be the Picard iteration association to Γ. Then there exist $\nu_k \to \nu^*$, $\lim_{k\to\infty} \Gamma^k\nu = \nu^*$, ν^* is a unique fixed point.

Proof: Let $\nu_0 \in \mathcal{H}$. We define the iterative sequence$\{\nu_k\}$ in $\boldsymbol{\mathcal{H}}$ defined as

$$\nu_{k+1} = \Gamma\nu_k = \Gamma^{k+1}\nu \text{ , for every } k \geq 1.$$

Now put $\nu = \nu_k$ and $\xi = \nu_{k-1}$ in (3.1.1) we have

$$\partial(\nu_{k+1},\nu_k) = \partial(\Gamma\nu_k,\Gamma\nu_{k-1})$$

$$\leq \theta_1 \frac{\partial(\nu_k,\Gamma\nu_k).\partial(\nu_{k-1},\Gamma\nu_{k-1})+\partial(\nu_k,\Gamma\nu_{k-1}).\partial(\nu_{k-1},\Gamma\nu_k)}{\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1})+\partial(\nu_k,\Gamma\nu_{k-1})+\partial(\nu_{k-1},\Gamma\nu_k)}$$

$$+\theta_2 \frac{\partial(\nu_k,\Gamma\nu_k).\partial(\nu_{k-1},\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1}).\partial(\nu_k,\Gamma\nu_{k-1})}{\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1})+\partial(\nu_k,\Gamma\nu_{k-1})}$$

$$+\theta_3 \frac{\partial(\nu_k,\Gamma\nu_k).\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_k,\Gamma\nu_{k-1}).\partial(\nu_{k-1},\Gamma\nu_k)}{\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_k,\Gamma\nu_{k-1})+\partial(\nu_{k-1},\Gamma\nu_k)}$$

$$+\theta_4 \frac{\partial(\nu_k,\Gamma\nu_k).\partial(\nu_k,\Gamma\nu_{k-1})+\partial(\nu_{k-1},\Gamma\nu_k).\partial(\nu_{k-1},\Gamma\nu_{k-1})}{\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_k,\Gamma\nu_{k-1})+\partial(\nu_{k-1},\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1})} + \theta_5\partial(\nu,\nu_{k-1}).$$

$$= \theta_1 \frac{\partial(\nu_k,\nu_{k+1}).\partial(\nu_{k-1},\nu_k)+\partial(\nu_k,\nu_k).\partial(\nu_{k-1},\nu_{k+1})}{\partial(\nu_k,\nu_{k+1})+\partial(\nu_{k-1},\nu_k)+\partial(\nu_k,\nu_k)+\partial(\nu_{k-1},\nu_{k+1})}$$

$$+\theta_2 \frac{\partial(\nu_k,\nu_{k+1}).\partial(\nu_{k-1},\nu_{k+1})+\partial(\nu_{k-1},\nu_k).\partial(\nu_k,\nu_k)}{\partial(\nu_k,\nu_{k+1})+\partial(\nu_{k-1},\nu_{k+1})+\partial(\nu_{k-1},\nu_k)+\partial(\nu_k,\nu_k)}$$

$$+\theta_3 \frac{\partial(\nu_k,\nu_{k+1}).\partial(\nu_k,\nu_{k+1})+\partial(\nu_k,\nu_k).\partial(\nu_{k-1},\nu_{k+1})}{\partial(\nu_k,\nu_{k+1})+\partial(\nu_k,\nu_{k+1})+\partial(\nu_k,\nu_k)+\partial(\nu_{k-1},\nu_{k+1})}$$

$$+\theta_4 \frac{\partial(\nu_k,\nu_{k+1}).\partial(\nu_k,\nu_k)+\partial(\nu_{k-1},\nu_{k+1}).\partial(\nu_{k-1},\nu_k)}{\partial(\nu_k,\nu_{k+1})+\partial(\nu_k,\nu_k)+\partial(\nu_{k-1},\nu_{k+1})+\partial(\nu_{k-1},\nu_k)} + \theta_5\partial(\nu,\nu_{k-1}).$$

$$\leq \left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2}\right)\partial(\nu_k,\nu_{k+1}) + \left(\frac{\theta_4}{2} + \theta_5\right)\partial(\nu_{k-1},\nu_k).$$

Therefore,

$$\partial(\nu_{k+1},\nu_k) \leq \pi\partial(\nu_{k-1},\nu_k).$$

Where $\pi = \frac{\left(\frac{\theta_4}{2}+\theta_5\right)}{\left(1-\frac{\theta_1}{2}-\frac{\theta_2}{2}-\frac{\theta_3}{2}\right)} < 1$, but $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\theta_5 < 2$.

So, continuing this process, we get

$$\partial(\nu_{k+1},\nu_k) \leq \pi^k\partial(\nu_0,\nu_1).$$

Now for $l > k$ and using the triangle inequality, we obtain

$$\partial(\nu_k,\nu_l) \leq \partial(\nu_k,\nu_{k+1}) + \partial(\nu_{k+1},\nu_{k+2}) + \cdots + \partial(\nu_{l-1},\nu_l)$$

$$\leq \pi^k\partial(\nu_0,\nu_1) + \pi^{k+1}\partial(\nu_0,\nu_1) + \cdots + \pi^{k+l-1}\partial(\nu_0,\nu_1)$$

$$\leq (\pi^k + \pi^{k+1} + \cdots + \pi^{k+l-1})\partial(\nu_0, \nu_1)$$

$$\leq \frac{\pi^k}{1-\pi}\partial(\nu_0, \nu_1)$$

Since Ω is normal cone with normal constant $\boldsymbol{\chi}$.So, we get

$\| \partial(\nu_k, \nu_l)\| \leq \chi \frac{\pi^k}{1-\pi} \|\partial(\nu_0, \nu_1)\| \to 0$ as $k \to \infty$. So, we have

$\partial(\nu_k, \nu_l) \to 0$, $k, l \to \infty$. Hence in $\boldsymbol{\mathcal{H}}$, $\{\nu_k\}$ is a Cauchy sequence. Due to the fact that $(\mathcal{H}, \partial)$ is a complete cone metric spaces, $\exists$ a point $\nu^* \in \mathcal{H}$ Such that $\nu_k \to \nu^*$ as $k \to \infty$. Again, since Γ is a continuous such that

$$\Gamma(\nu) = \Gamma(\lim_{k\to\infty} \nu_k) = \lim_{k\to\infty} \Gamma\nu_k = \lim_{k\to\infty} \nu_{k+1} = \nu^*.$$

Hence ν^* is fixed point of Γ in $\boldsymbol{\mathcal{H}}$.

Now we will show that ν^* is a unique fixed point of Γ in $\boldsymbol{\mathcal{H}}$.

Suppose ξ^* is another fixed point of Γ in $\boldsymbol{\mathcal{H}}$ such that $\Gamma\xi^* = \xi^*$. Then we have

$$\partial(\nu^*, \xi^*) = \partial(\Gamma\nu^*, \Gamma\xi^*)$$

$$\leq \theta_1 \frac{\partial(\nu^*,\Gamma\nu^*).\partial(\xi^*,\Gamma\xi^*)+\partial(\nu^*,\Gamma\xi^*).\partial(\xi^*,\Gamma\nu^*)}{\partial(\nu^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*)+\partial(\nu^*,\Gamma\xi^*)+\partial(\xi^*,\Gamma\nu^*)} + \theta_2 \frac{\partial(\nu^*,\Gamma\nu^*).\partial(\xi^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*).\partial(\nu^*,\Gamma\xi^*)}{\partial(\nu^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*)+\partial(\nu^*,\Gamma\xi^*)}$$

$$+ \theta_3 \frac{\partial(\nu^*,\Gamma\nu^*).\partial(\nu^*,\Gamma\nu^*)+\partial(\nu^*,\Gamma\xi^*).\partial(\xi^*,\Gamma\nu^*)}{\partial(\nu^*,\Gamma\nu^*)+\partial(\nu^*,\Gamma\nu^*)+\partial(\nu^*,\Gamma\xi^*)\partial(\xi^*,\Gamma\nu^*)} + \theta_4 \frac{\partial(\nu^*,\Gamma\nu^*).\partial(\nu^*,\Gamma\xi^*)+\partial(\xi^*,\Gamma\nu^*).\partial(\xi^*,\Gamma\xi^*)}{\partial(\nu^*,\Gamma\nu^*).\partial(\nu^*,\Gamma\xi^*)+\partial(\xi^*,\Gamma\nu^*).\partial(\xi^*,\Gamma\xi^*)}$$

$$+ \theta_5 \partial(\nu^*, \xi^*).$$

$$= \theta_1 \frac{\partial(\nu^*,\nu^*).\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*).\partial(\xi^*,\nu^*)}{\partial(\nu^*,\nu^*)+\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*)+\partial(\xi^*,\nu^*)} + \theta_2 \frac{\partial(\nu^*,\nu^*).\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*).\partial(\nu^*,\xi^*)}{\partial(\nu^*,\nu^*)+\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*)}$$

$$+ \theta_3 \frac{\partial(\nu^*,\nu^*).\partial(\nu^*,\nu^*)+\partial(\nu^*,\xi^*).\partial(\xi^*,\nu^*)}{\partial(\nu^*,\nu^*)+\partial(\nu^*,\nu^*)+\partial(\nu^*,\xi^*)\partial(\xi^*,\nu^*)} + \theta_4 \frac{\partial(\nu^*,\nu^*).\partial(\nu^*,\xi^*)+\partial(\xi^*,\nu^*).\partial(\xi^*,\xi^*)}{\partial(\nu^*,\nu^*).+\partial(\nu^*,\xi^*)+\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*)}$$

$$+ \theta_5 \partial(\nu^*, \xi^*).$$

Thus

$$\|\partial(\nu^*, \xi^*)\| \leq \chi\left[\frac{\boldsymbol{\theta_1}}{\boldsymbol{2}}\|\partial(\nu^*, \xi^*)\| + \frac{\boldsymbol{\theta_3}}{\boldsymbol{2}}\|\partial(\nu^*, \xi^*)\| + \theta_5\|\partial(\nu^*, \xi^*)\|.\right]$$

$\leq \chi\left[\frac{\boldsymbol{\theta_1}}{\boldsymbol{2}} + \frac{\boldsymbol{\theta_3}}{\boldsymbol{2}} + \frac{\boldsymbol{\theta_4}}{\boldsymbol{2}} + \theta_5\right]\|\partial(\nu^*, \xi^*)\| \to 0$, which is a contradiction, because $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\theta_5 < 2$. So, $\|\partial(\nu^*, \xi^*)\| = \mathbf{0}$ .Implies that $\nu^* = \xi^*$, thus,ν^* is a unique fixed point of Γ in $\boldsymbol{\mathcal{H}}$.

Example 3.2.: Let $\boldsymbol{\mathcal{H}}$= [0, 1] with partial order " $\leq$ " and let $\Gamma: \mathcal{H} \to \mathcal{H}$ be defined by

$$\Gamma(\nu) = \begin{cases} \frac{1}{2} & \text{if} \in [0, \frac{1}{4}) \\ \nu - \frac{1}{4} & \text{if } \nu \in [\frac{1}{4}, 1) \end{cases}$$

To show that Γ satisfies the contractive condition (3.1.1) of theorem (3.1), for $=\frac{1}{2}$, $\xi=\frac{1}{6}$, $\theta_1=\theta_2=\theta_3=\theta_4=1$. Then we have

$\Gamma\nu=\frac{1}{4}$, $\Gamma\xi=\frac{1}{2}$, $\partial(\nu,\Gamma\nu)=\frac{1}{4}$, $\partial(\xi,\Gamma\xi)=\frac{1}{3}$, $\partial(\nu,\Gamma\xi)=0$, $\partial(\xi,\Gamma\nu)=\frac{1}{12}$, $\partial(\nu,\xi)=\frac{1}{3}$ and

$\partial(\Gamma\nu,\Gamma\xi)=\frac{1}{4}$.

Thus, from (3.1.1), we get

$$\begin{aligned}\frac{1}{4} &= \partial(\Gamma\nu,\Gamma\xi)\\ &\leq \frac{2}{3}+\frac{1}{32}+\frac{3}{28}+\frac{1}{24}+\frac{1}{3}\theta_5.\\ &= \frac{448+21+72+28}{672}+\frac{1}{3}\theta_5\\ \frac{1}{4} &= \frac{569}{672}+\frac{1}{3}\theta_5.\end{aligned}$$

It follows that, $\theta_5\geq\frac{401}{672}$. Hence all the conditions of Theorem 3.1 are satisfied and Γ has a unique fixed point $\nu=\frac{1}{2}$.

Theorem 3.2.: Let us Consider that, $(H,\leq)$ be a partially ordered set and let there is a cone metric ∂ on $\mathcal{H}$ such that χ be a normal constant with normal cone Ω on complete cone metric space$(\mathcal{H},\partial)$. Assume that, $\Gamma:\mathcal{H}\to\mathcal{H}$ be a continuous self mapping such that $\nu,\xi\in\mathcal{H}$, $\nu\neq\xi$ there exist $\theta_1,\theta_2,\theta_3,\theta_4,\in[0,1)$ with $2\theta_1+2\theta_2+\theta_4<2$, satisfies the following condition

$$\partial(\Gamma\nu,\Gamma\xi)\leq\theta_1\partial(\nu,\xi)+\theta_2\frac{\partial(\nu,\Gamma\nu).\partial(\xi,\Gamma\xi)+\partial(\nu,\Gamma\xi).\partial(\xi,\Gamma\nu)}{\partial(\nu,\xi)}+\theta_3\frac{\partial(\nu,\Gamma\nu)[.\partial(\nu,\Gamma\nu)+\partial(\xi,\Gamma\xi)]}{\partial(\nu,\xi)+\partial(\xi,\Gamma\nu)+\partial(\xi,\Gamma\xi)}$$
$$+\theta_4\frac{\partial(\nu,\Gamma\nu).\partial(\xi,\Gamma\nu)+\partial(\xi,\Gamma\xi).\partial(\nu,\Gamma\xi)}{\partial(\nu,\Gamma\nu)+\partial(\xi,\Gamma\nu)+\partial(\xi,\Gamma\xi)+\partial(\nu,\Gamma\xi)}. \qquad (3.2.1)$$

For $\nu_0\in\mathcal{H}$, suppose $\{\nu_k\}_{k=0}^{\infty}\subset\mathcal{H}$ defined by $\nu_{k+1}=\Gamma\nu_k, k=0,1,2\,....$ be the Picard iteration association to Γ. Then there exist $\nu_k\to\nu^*$, $\lim_{k\to\infty}\Gamma^k\nu=\nu^*$, ν^* is a unique fixed point.

Proof: Let $\nu_0\in\mathcal{H}$ and define the iterative sequence$\{\nu_k\}$ in $\mathcal{H}$such that

$$\nu_{k+1}=\Gamma\nu_k=\Gamma^{k+1}\nu \text{ , for every } k\geq 1.$$

Now put $\nu=\nu_k$ and $\xi=\nu_{k-1}$ in (3.2.1) we have

$$\begin{aligned}\partial(\nu_{k+1},\nu_k)&=\partial(\Gamma\nu_k,\Gamma\nu_{k-1})\\ &\leq\theta_1\partial(\nu_k,\nu_{k-1})+\theta_2\frac{\partial(\nu_k,\Gamma\nu_k).\partial(\nu_{k-1},\Gamma\nu_{k-1})+\partial(\nu_k,\Gamma\nu_{k-1}).\partial(\nu_{k-1},\Gamma\nu_k)}{\partial(\nu,\nu_{k-1})}\\ &+\theta_3\frac{\partial(\nu_k,\Gamma\nu_{k-1})[\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1})]}{\partial(\nu_k,\nu_{k-1})+\partial(\nu_k-1,\Gamma\nu_{k-1})+\partial(\nu_{k-1},\Gamma\nu_k)}\\ &+\theta_4\frac{\partial(\nu_k,\Gamma\nu_k).\partial(\nu_{k-1},\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1}).\partial(\nu_k,\Gamma\nu_{k-1})}{\partial(\nu_k,\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_k)+\partial(\nu_{k-1},\Gamma\nu_{k-1})+\partial(\nu_k,\Gamma\nu_{k-1})}.\end{aligned}$$

$$= \theta_1\partial(\nu_k,\nu_{k-1}) + \theta_2 \frac{\partial(\nu_k,\nu_{k+1}).\partial(\nu_{k-1},\nu_k)+\partial(\nu_k,\nu_k).\partial(\nu_{k-1},\nu_{k+1})}{\partial(\nu,\nu_{k-1})}$$

$$+ \quad \theta_3 \frac{\partial(\nu_k,\nu_k)[\partial(\nu_k,\nu_{k+1})+\partial(\nu_{k-1},\nu_k)]}{\partial(\nu_k,\nu_{k-1})+\partial(\nu_{k-1},\nu_k)+\partial(\nu_{k-1},\nu_{k+1})}$$

$$+ \theta_4 \frac{\partial(\nu_k,\nu_{k+1}).\partial(\nu_{k-1},\nu_{k+1})+\partial(\nu_k,\nu_k).\partial(\nu_{k-1},\nu_{k+1})}{\partial(\nu_k,\nu_{k+1})+\partial(\nu_{k-1},\nu_{k+1})+\partial(\nu_{k-1},\nu_k)+\partial(\nu_k,\nu_k)}.$$

$$\leq \theta_1\partial(\nu_k,\nu_{k-1}) + (\theta_2 + \frac{\theta_4}{2})\partial(\nu_k,\nu_{k-1}).$$

Therefore,

$$\partial(\nu_{k+1},\mathrm{v}_\mathrm{k}) \leq \Psi\partial(\nu_k,\nu_{k-1}).$$

Where $\Psi = \frac{\theta_1}{\left(1-\theta_2-\frac{\theta_4}{2}\right)} < 1$, but $2\theta_1 + 2\theta_2 + \theta_4 < 2$.

So, continuing this process, we get

$$\partial(\nu_{k+1},\mathrm{v}_\mathrm{k}) \leq \Psi^k\partial(\nu_0,\nu_1).$$

Now for $l > k$ and using the triangle inequality, we obtain

$$\partial(\nu_k,\nu_l) \leq \partial(\nu_k,\nu_{k+1}) + \partial(\nu_{k+1},\nu_{k+2}) + \cdots + \partial(\nu_{l-1},\nu_l)$$

$$\leq \Psi^k\partial(\nu_0,\nu_1) + \Psi^{k+1}\partial(\nu_0,\nu_1) + \cdots + \Psi^{k+l-1}\partial(\nu_0,\nu_1)$$

$$\leq (\Psi^k + \Psi^{k+1} + \cdots + \Psi^{k+l-1})\partial(\nu_0,\nu_1)$$

$$\leq \frac{\Psi^k}{1-\Psi}\partial(\nu_0,\nu_1)$$

Since Ω is normal cone with normal constant $\boldsymbol{\chi}$.So, we get

$\| \partial(\nu_k,\nu_l)\| \leq \chi\frac{\Psi^k}{1-\Psi} \quad \|\partial(\nu_0,\nu_1)\| \to 0$ as $k \to \infty$. So, we have

$\partial(\nu_k,\nu_l) \to 0$, $k,l \to \infty$. Hence in $\boldsymbol{\mathcal{H}}$, $\{\nu_k\}$ is a Cauchy sequence. Due to the fact that $(\mathcal{H},\partial)$ is a complete cone metric spaces, $\exists\ \nu^* \in \mathcal{H}$ Such that $\nu_k \to \nu^*$ as $k \to \infty$. Again, since Γ is a continuous such that

$$\Gamma(\nu) = \Gamma(\lim_{k\to\infty}\nu_k) = \lim_{k\to\infty}\Gamma\mathrm{v}_\mathrm{k} = \lim_{k\to\infty}\nu_{k+1} = \nu^*.$$

Hence ν^* is fixed point of Γ in $\boldsymbol{\mathcal{H}}$.

Now we will show that ν^* is a unique fixed point of Γ in $\boldsymbol{\mathcal{H}}$.

Suppose ξ^* is another fixed point of Γ in $\boldsymbol{\mathcal{H}}$ such that $\Gamma\xi^* = \xi^*$. Then we have

$$\partial(\nu^*,\xi^*) = \partial(\Gamma\nu^*,\Gamma\xi^*)$$

$$\leq \theta_1\partial(\nu^*,\xi^*) + \theta_2\frac{\partial(\nu^*,\Gamma\nu^*).\partial(\xi^*,\Gamma\xi^*)+\partial(\nu^*,\Gamma\xi^*).\partial(\xi^*,\Gamma\nu^*)}{\partial(\nu^*,\xi^*)} + \theta_3\frac{\partial(\nu^*,\Gamma\nu^*)[.\partial(\nu^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*)]}{\partial(\nu^*,\xi^*)+\partial(\xi^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*)}$$

$$+ \theta_4\frac{\partial(\nu^*,\Gamma\nu^*).\partial(\xi^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*).\partial(\nu^*,\Gamma\xi^*)}{\partial(\nu^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\nu^*)+\partial(\xi^*,\Gamma\xi^*)+\partial(\nu^*,\Gamma\xi^*)}.$$

$$\leq \theta_1\partial(\nu^*,\xi^*) + \theta_2\frac{\partial(\nu^*,\nu^*).\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*).\partial(\xi^*,\nu^*)}{\partial(\nu^*,\xi^*)} + \theta_3\frac{\partial(\nu^*,\nu^*)[.\partial(\nu^*,\nu^*)+\partial(\xi^*,\xi^*)]}{\partial(\nu^*,\xi^*)+\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*)}$$

$$+ \theta_4\frac{\partial(\nu^*,\nu^*).\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*).\partial(\nu^*,\xi^*)}{\partial(\nu^*,\nu^*)+\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*)}.$$

Thus

$\| \partial(\nu^*,\xi^*)\| \leq \chi[(\theta 1+\theta_2)\| \partial(\nu^*,\xi^*)\| \to 0$. which is a contradiction, because $2\theta_1+2\theta_2+\theta_4 < 2$, So, $\|\partial(\nu^*,\xi^*)\| = \mathbf{0}$ .Implies that $\nu^* = \xi^*$, thus,ν^* is a unique fixed point of Γ in $\mathcal{H}$.

Example 3.2.: Let $\mathcal{H}= \left\{\frac{1}{4},\frac{1}{2},1\right\}$with partial order " ≤ " and $\partial(\nu,\xi) = |\nu-\xi|$ be a cone metric space on $\mathcal{H}$. Assume that $\Gamma:\mathcal{H}\to\mathcal{H}$ be defined by

$$\Gamma(\nu) = \begin{cases} \nu^2 \ , & \text{if} \in [0,\frac{1}{2}] \\ 1-\nu \ , & \text{if } \nu \in (\frac{1}{2}, 1] \\ \frac{\nu}{2} \ , & \text{if } \nu \in (1,\infty) \end{cases}$$

To show that Γ satisfies the contractive condition (3.1.1) of theorem (3.1), for $=\frac{1}{2}$, $\xi = 1$ $\theta_2 = \theta_3 = \theta_4 = 1$. Then we have

$$\Gamma\nu = \frac{1}{4}, \Gamma\xi = \frac{1}{2}, \partial(\nu,\Gamma\nu) = \partial\left(\frac{1}{2},\Gamma\frac{1}{2}\right) = \frac{1}{4},$$

$$\partial(\xi,\Gamma\xi) = \partial(1,\Gamma 1) = \frac{1}{2},$$

$$\partial(\nu,\Gamma\xi) = \partial\left(\frac{1}{2},\Gamma 1\right) = 0,$$

$$\partial(\xi,\Gamma\nu) = \partial\left(1,\Gamma\frac{1}{2}\right) = \frac{3}{4},$$

$$\partial(\nu,\xi) = \partial\left(\frac{1}{2},1\right) = \frac{1}{2} \text{ and } \quad \partial(\Gamma\nu,\Gamma\xi) = \partial\left(\Gamma\frac{1}{2},\Gamma 1\right) = \frac{1}{4}.$$

Thus, from (3.1.1), we get

$$\begin{aligned} \frac{1}{4} &= \partial(\Gamma\nu,\Gamma\xi) \\ &\leq \frac{1}{2}\theta_1 + \frac{1}{4} + 0 + \frac{1}{8} \\ &= \frac{1}{2}\theta_1 + \frac{3}{8}. \end{aligned}$$

It follows that, $\theta_1 \leq \frac{1}{4}$. Hence all the conditions of Theorem 3.1 are satisfied and Γ has a unique fixed point $\nu = \frac{1}{4}$.

4. Conclusion

In this article, we have obtained some fixed point theorems for self-mapping satisfying rational type contractive conditions in complete cone metric space by using normality of cone, which is a generalization and extension of the results due to R. Bhardwaj et al.[7] from metric space to cone metric space and validates these results with an example.

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